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Some Special Seminear-ring Structures II

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Abstract

In this paper we introduce the concepts of P_k and P_k' seminear-rings where *'k'* is a positive integer and study some of their properties. We also discuss certain properties of P_1 and P_1' seminear-rings and also obtain their characterizations.

Key words: P_k and P_k' seminear-ring, Mate function, Ideal, Idempotent, Nilpotent.

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1.Introduction

This paper is a continuation of the authors' earlier paper[6]. In the previous paper[6] properties of $P(r,m)$ seminear-rings have been discussed. The purpose of the present paper is to introduce the concept of P_k and P_k' seminear-rings. By a seminear-ring *'R'* we mean only **a right seminear-ring** $(R, +, .)$ with an absorbing zero as defined[6]. We write ab to denote the product $a.b$ for any two elements a, b in R. For terms and notations used but left undefined we refer to[1-3].

We recall[6] that a function $f: R \to R$ is a mate function if $x = xf(x)x$ for all x in R.

1.1. *Notations :*

(i) $E=\{e\in R/e^2=e\}$-set of all idempotents of R.
(ii) $C(R) = \{r\in R \,/\, rx = xr$ for all $x\in R\}$- centre of R.
(iii) $L=\{x\in R \,/\, x^k = 0$ for some positive integer 'k'$\}$- set of all nilpotent elements of R.

1.2. *Preliminary results:*

We freely make use of the following results from[6] and designate them as K(1), K(2), K(3) and K(4).

K(1): A seminear-ring R has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in R. (This result in prob[5] 14, P.9 in respect of rings is valid for R as well).

K(2): If R admits a mate function f, then $xf(x)$, $f(x)x \in E$ and $Rx = Rf(x)x$ and $xR = xf(x)R$ for all x in R (Proposition[6] 3.2).

K(3): A mate function 'f' of R is called a P_3 mate function if for every x in R, $xf(x) = f(x)x$ (Definition[6] 4.2)

K(4): If R is a $P(1,2)$ seminear-ring *i.e.* $xR = Rx^2$ for all $x \in R$ then $E \subseteq C(R)$ (Theorem[6] 4.18).

2. P_k *and* P_k' *seminear-ring* :

In this section we define P_k and P_k' seminear-rings and give certain examples of such seminear-rings.

Definition 2.1. A seminear-ring R is called a P_k seminear-ring (P_k' seminear-ring) if there exists a positive integer 'k' such that $x^kR = xRx$ ($Rx^k = xRx$) for all x in R.

Examples 2.2.

(i) Let R = $\{0, a, b, c, d\}$. We define the semigroup operations "+" and "." in R as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	c
d	d	a	b	c	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
d	0	a	b	c	d

Then $(R, +, .)$ is a P_k as well as a P_k' seminear-ring for all positive integer k.

(ii) Let $R = \{0, a, b, c, d\}$. We define the semigroup operations "+" and "." in R as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	d	d	d

Then $(R, +, .)$ is a P_k seminear-ring for all positive integers k but not a P_k' seminear-ring for any positive integer k.

(iii) The direct product[4,5] of any two seminear-fields is a P_k as well as a P_k' seminear-ring.

Theorem 2.3. Any homomorphic image of a $P_k(P_k')$ seminear-ring is a $P_k(P_k')$ seminear-ring.

Proof. The proof is straight forward.

Similar to the definitions of left normal and right normal seminear-ring (Definition[6] 2.5) we define the following:

Definition 2.4. Let *'r'* be a positive integer. We say that R is a left-r-normal (right-r-normal) seminear-ring if $a \in Ra^r (a \in a^r R)$ for all *'a'* in R.

Example 2.5.

(a) The seminear-ring of example 2.2(i) is a left-r-normal as well as a right-r-normal seminear-ring.
(b) Trivially any Boolean seminear-ring is a left-r-normal as well as a right-r-normal seminear-ring.

Proposition 2.6. Every left-r-normal (right-r-normal) seminear-ring is a left (right) normal seminear-ring.

Proof. Let R be a left-r-normal seminear-ring with $r \geq 2$. Clearly then for all $a \in R$, $a \in Ra^r = (Ra^{r-1})a \subseteq Ra$. i.e. $a \in Ra$. Therefore R is a left normal seminear-ring.

Proof is similar when R is a right-r-normal seminear-ring.

We shall now discuss some elementary properties of $P_k(P_k')$ seminear-ring.

Proposition 2.7. A left identity (right identity) of a $P_k(P_k')$ seminear-ring is also a right identity(left identity).

Proof. Let R be a P_k seminear-ring. Let *'e'* be a left identity of R. Then $x = ex$ for all $x \in R$. Now $e^k R = eRe \Rightarrow eR = eRe$. Then there exists $y \in R$ such that, $x = ex = eye = (ey)e = ye$. Hence $xe = (ye)e = ye^2 = ye = x$. i.e. $x = ex = xe$. Therefore *'e'* is a right identity as well.

Let R be a P_k' seminear-ring and *'e'* be a right identity of R. Then $x = xe$ for all $x \in R$. Now $Re^k = eRe \Rightarrow Re = eRe$. Then there exists $y' \in R$ such that, $x = xe = ey'e = e(y'e) = ey'$. Hence $ex = e(ey') = e^2 y' = ey' = x$. i.e. $x = xe = ex$. It follows that *'e'* is also a left identity.

Remark 2.8. A right identity[8,9] of a P_k seminear-ring need not be a left identity. The following example substantiates this. We consider the seminear-ring $R = \{0, a, b, c, d\}$ where the semigroup operations ''+'' and ''.'' in R are defined as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	b
c	0	c	c	c	c
d	0	d	d	d	d

Here a, b, c, d are right identities but none is a left identity.

Theorem 2.9. A P_k (P_k') seminear-ring R has a mate function if and only if R is a right-k-normal (left-k-normal) seminear-ring.

Proof. Let R be a P_k seminear-ring. Then $x^kR = xRx$ for all x in R. If R has a mate function 'f' then $x = xf(x)x \in xRx\ (= x^kR)$ and this implies $x \in x^kR$. *i.e.* R is a right-k-normal seminear-ring.

Conversely let R be a right-k-normal P_k seminear-ring. Therefore $x \in x^kR\ (= xRx)$ for all x in R. Then there exists some y in R such that $x = xyx$. Clearly then $x = xf(x)x$ where we set $f(x)=y$. It follows that f is a mate function for R.

The proof is similar when R is a P_k' seminear-ring.

Theorem 2.10. Let R be a P_k or a P_k' seminear-ring. If R admits mate functions then R has no non-zero nilpotent elements i.e. L= {0}.

Proof. Let R admit a mate function 'f '. We shall show that $x^2 = 0 \Rightarrow x = 0$ for x in R ....(1)

Case (i): Let R be a P_1 seminear-ring, i.e. $xR= xRx$ for all x in R. We have $x = xf(x)x\in xRx$. But $xRx= (xR)x = (xRx)x = xRx^2= (xR)x^2$. Then there exists $y\in R$ such that $x = xyx^2$. Consequently (1) holds.

Case (ii): Let R be a P_k seminear-ring with k >1. Now $x^kR = xRx$ for all x in R. Since $x = xf(x)x\in xRx = x^kR$, $x = x^ky$ for some 'y' in R. If k = 2, then $x = x^2y$. If $k > 2$, we write $x = x^2\ (x^{k-2}y)$ and therefore (1) is true.

Case (iii): Let R be a P_1' seminear-ring, *i.e.* $Rx = xRx$ for all x in R. Therefore $x = xf(x)x \in xRx = x(Rx) = x(xRx) = x^2Rx \Rightarrow x = x^2(Rx)$. Then there exists $y\in R$ such that $x = x^2yx$. Thus (1) holds good.

Case (iv): Let R be a P_k' seminear-ring with k >1. Now $Rx^k = xRx$ for all x in R. Since $x = xf(x)x \in xRx = Rx^k$, we get $x = y2\ x^k$ for some $y2$ in R. If k = 2, then $x = y'x^2$. If $k > 2$, we write $x = (y'x^{k-2})\ x^2$ and again (1) holds.

Now K(1) gurantees that, in all the four cases, L = {0}.

Theorem 2.11. Let R be a $P(1,2)$ seminear-ring with a mate function f . Then R is a

(a) P_k seminear-ring for all positive integer k.

(b) P_k' seminear-ring for all positive integer k.

Proof. Since R is a $P(1,2)$ seminear-ring. K(4) demands that every idempotent is central. i.e $E \subseteq C(R)$.

(a) *Case (i):* Let k = 1. For all x in R, $xR = x(f(x)xR) = x(Rf(x)x)$ (since $E \subseteq C(R)$) $= xRx$ (By K(2)) i.e. $xR = xRx$. Hence R is a P_1 seminear-ring.

Case (ii): For k >1 and for any $x\in R$, $x^kR = x(x^{k-1}\ R) \subseteq xR = xRx$ (using the result for k=1). Therefore $x^kR \subseteq xRx$. Also $xRx = xRxf(x)x = x(Rxf(x))x = x(xf(x)R)x$ (since $E \subseteq C(R)$) $= x(xR)x$ (By K(2)) $= x^2R = x(xRx) = x(x^2Rx) = x^3Rx$. Repeating this process, we obtain $xRx = x^kRx \subseteq x^kR$ for all positive integers

k .Therefore $xRx \subseteq x^kR$. Thus $xRx = x^kR$ for all x in R. Hence R is P_k seminear-ring for any positive integer k.

(b) *Case(i):* Let $k = 1$. For all x in R, $Rx = Rxf(x)x = (Rxf(x))x = (xf(x)R)x$ (since E $\subseteq$ C(R)) $= xRx$ (By K(2)). i.e. $Rx= xRx$. Hence R is a $P_1{}'$ seminear-ring.

Case (ii): Let $k >1$. Since $E \subseteq C(R)$ we have for all y,x in R, $yx^k = (yx)x^{k-1} = (yxf(x)x)x^{k-1} = (xf(x)yx)x^{k-1} = (xf(x)yx^{k-1})x \in xRx$. Therefore $Rx^k \subseteq xRx$. Also $xyx = (xf(x)x)yx = xyf(x)x^2 = (xf(x)x)yf(x)x^2 = xy(f(x))^2x^3$. Repeating this process, we obtain $xyx = xy(f(x))^{k-1}x^k \in Rx^k$ for all positive integers k. Therefore $xRx \subseteq Rx^k$. Thus $xRx = Rx^k$ for all x in R. Hence R is P_k' seminear-ring for any positive integer k.

Remark 2.12. We observe that a P_k seminear-ring need not be a $P(1,2)$ seminear-ring. For example, we consider the seminear-ring $R=\{0, a, b, c, d\}$ where the semigroup operations ‘’+’’ and ‘’.’’ in R are defined as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	d
d	0	a	d	d	d

Here $(R, +, .)$ is a P_k seminear-ring for all positive integers k. Even though the identity function serves as a mate function for R, it is not a $P(1, 2)$ seminear-ring.

Proposition 2.13. Let R admit a P_3 mate function ‘f ‘. Then every right ideal (left ideal) of a P_k $(P_k{}')$ seminear-ring R is a completely semi prime ideal.

Proof. Case (i): If $k = 1$. Let I be a right ideal of R and let $a^2 \in I$. Then $a = af(a)a = a(f(a)a) = a(af(a))$ (since f is a P_3 mate function) $= a^2f(a) \in IR \subseteq I$. i.e. $a \in I$ and the result follows.

Case (ii): Let $k >1$. For $a \in R$, $a = af(a)a \in aRa = (a^kR)$ and therefore there exists $y \in R$ such that $a = a^ky$. When $k = 2$, $a^2 \in I \Rightarrow a = a^2y \in IR \subseteq I$. i.e. $a^2 \in I \Rightarrow a \in I$. When $k>2$, $a^2 \in I \Rightarrow a = a^2(a^{k-2}y) \in IR \subseteq I$. i.e. $a \in I$ and the desired result follows.

The proof is similar when R is a P_k’ seminear-ring.

Theorem 2.14. Any ideal of an left-k-normal P_k’ (right-k-normal P_k) seminear-ring R is also an left-k-normal P_k’ (right-k-normal P_k) seminear-ring in its own right.

Proof. Since R is a left-k-normal P_k’ seminear-ring Proposition 2.9 guarantees the existence of a mate function f for R. Let M be an ideal of R. Therefore $f(x)xf(x) \in RMR \subseteq M$ for all x in M. Thus we can define a map g: $M \to M$ such that $g(x) = f(x)xf(x)$ for all $x \in M$. Obviously then $xg(x)x = x$ and therefore g is a mate function for M.

Now let x, $a \in M$. Since $Rx^k = xRx$ there exists $b \in R$ such that $ax^k = xbx = x(bxg(x))x \in x(RM)x \subseteq xMx$. Therefore $Mx^k \subseteq xMx$ ....(1) .

Also since $xax \in xRx = Rx^k$, there exists $y \in R$ such that $xax = yx^k$. Again $xax = xg(x)(xax) = xg(x)yx^k = y2\ x^k$ where $y2 = xg(x)y \in MR \subseteq M$. Therefore $xMx \subseteq Mx^k$............(2).

From (1) and (2) we get $Mx^k = xMx$ for all $x \in M$. i.e. M is a P_k' seminear-ring. Since M has a mate function 'g' then M is a left-k-normal seminear-ring as well (from Proposition 2.9).

The proof is similar when R is a right-k-normal P_k seminear-ring.

Definition[7] *2.15.* A seminear-ring R is said to fulfill the right(left) Ore condition with respect to a given subsemigroup 'A' of $(R, .)$ if for every $a \in A$, $r \in R$ there exist $a_1 \in A$, $r_1 \in R$ such that $ra_1 = ar_1 (a_1 r = r_1 a)$.

Proposition 2.16. Let R be a $P_k (P_k')$ seminear-ring. Then R satisfies left (right) Ore condition.

Proof. Let A be any subsemigroup of R and let $a \in A$, $r \in$ R. Since $a^k R = aRa$ there exists $y \in R$ such that $a^k r = aya$. i.e. $a_1 r = ar_1$ where $a_1 = a^k \in A$ and $r_1 = ya \in R$ and R fulfills the left Ore condition.

In a similar fashion we can prove that the P_k' seminear-ring fulfills the right Ore condition.

3. Properties of P_1 and P_1' seminear-rings:

We furnish below simple characterizations of P_1 and P_1' seminear-ring.

Theorem 3.1. Let R be a seminear-ring with a mate function f. Then we have

(i) every left ideal of R is a right ideal of R if and only if R is a P_1 seminear-ring.

(ii) every right ideal of R is a left ideal of R if and only if R is a P_1' seminear-ring.

Proof. Since R is a seminear-ring with a mate function f.

(i). Assume that every left ideal of R is a right ideal of R. By the assumption, Rx being a left ideal for every $x \in R$, is also a right ideal of R. Therefore $(Rx)R \subseteq Rx$. Since f is a mate function $x = xf(x)x$. From this we get $xR = xf(x)xR \in xRxR \subseteq xRx$........(1).

Clearly $xRx \subseteq xR$..........(2). From (1) and (2) we get $xR = xRx$ for all $x \in R$. i.e. R is a P_1 seminear-ring.

Conversely, let A be any left ideal of R, then $RA \subseteq A$. Let $a \in A$ and $y \in R$, we have $ay \in aR = aRa \Rightarrow ay = ay'a$ (for some y' in R) $= (ay')a \in Ra$. This forces $ay \in RA \subseteq A \Rightarrow AR \subseteq A$ and hence A is an ideal.

(ii). Assume that every right ideal of R is a left ideal of R. By the assumption xR, being a right ideal for every $x \in R$, is also a leftt ideal of R. Therefore $R(xR) \subseteq xR$. Since f is a mate function $x = xf(x)x$. From this we get $Rx = Rxf(x)x \in RxRx \subseteq xRx$........(1).

Clearly $xRx \subseteq Rx$..........(2). From (1) and (2) we get $xR = xRx$ for all $x \in R$. i.e. R is a P_1' seminear-ring.

Conversely, let A be a right ideal of R, then $AR \subseteq A$. Let $a \in A$ and $y \in R$, we have $ya \in Ra = aRa \Rightarrow ya = ay'a$, for some y' in $R = a(y'a) \in aR$. This forces $ya \in AR \subseteq A \Rightarrow RA \subseteq A$. Hence A is an ideal.

Definition[7] *3.2.* A seminear-ring R is said to have Insertion of Factors Property-IFP for short - if for $x, y \in R$, $xy = 0 \Rightarrow xry = 0$ for all r in R. If in addition, $xy = 0 \Rightarrow yx = 0$ for x, y in R we say R has ($*$, IFP).

Theorem 3.3. Let R be a left normal P_1' seminear-ring. Then

(i) $(M \cap N) = MN$ where M and N are ideals of R

(ii) Any prime ideal is a completely prime ideal.

(iii) R has ($*$, IFP)

Proof. Since R is left normal P_1' seminear-ring. Proposition 2.9 guarantees that R has a mate function f.

(i). If M, N are ideals of R then $(M \cap N)^2 = (M \cap N)(M \cap N) \subseteq M \cap N$. Also for all '$a$' in $M \cap N$, $a = a(f(a)a) \in (M \cap N)(M \cap N)$. This forces $(M \cap N) = (M \cap N)^2$. Further, $(M \cap N) = (M \cap N)(M \cap N) \subseteq MN$.

To prove the reverse inclusion, let us take $y \in MN$. Clearly then $y \in MN \subseteq N$. Also $y = xx'$ for some x in M and x' in N. This demands that $y \in xR$. Hence $y \in xR \subseteq MR \subseteq M$. Thus $y \in M \cap N$ and the desired result follows.

(ii). Let P be a prime ideal of R and let $ab \in P$. Therefore $Rab \subseteq RP \subseteq P$.

Since Ra and Rb are ideals of R. Then $Ra \cap Rb = RaRb$ (using the result(i)).

Also $Ra = Ra \cap R = RaR$. Hence $Rab = RaRb = Ra \cap Rb$.

This yields, $RaRb = (Rab) \subseteq P$ and since P is prime, $Ra \subseteq P$ or $Rb \subseteq P$. Therefore $(a =)\ af(a)a \in P$ or $(b =)\ bf(b)b \in P$ and the desired result follows.

(iii). Since R has a mate function 'f'. Theorem 2.10 guarantees that R has no non-zero nilpotent elements. If $xy = 0$ then $(yx)^2 = (yx)(yx) = y(xy)x = 0$. This implies $yx = 0$. Again for all $r \in R$, $(xry)^2 = (xry)(xry) = xr(yx)ry = 0$. Therefore $xry = 0$. Consequently R has ($*$, IFP).

We conclude this paper with the follwing Remark.

Remark 3.4. We observe, in view of Theorem 2.9, that the three results in Theorem 3.8 hold good for a right normal P_1 seminear-ring.

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