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A New approach for solving fuzzy assignment problem

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Abstract

Assignment problem is a well known topic and is used very often in solving problems of engineering and management sciences. If the cost is not deterministic, then the problem is said to be assignment problem with fuzzy costs. We propose a new algorithm to solve this fuzzy assignment problem.

Key words: Fuzzy numbers, fuzzy ranking.

Introduction

An assignment problem plays an important role in industry and other applications. In an assignment problem 'n' jobs are to be performed by 'n' persons depending upon their efficiency to do the job. We assume that one person can be assigned exactly one job; also each person can do at most one job. The main objective of assignment problem is to find the optimum assignment so that the total cost of performing all jobs is minimum or the total profit is maximum.

In this paper, we investigate fuzzy assignment problem (i.e.) assignment problem with fuzzy costs $\overline{C_{ij}}$. Our idea is to solve the fuzzy assignment problem without changing the fuzzy costs in to crisps costs. Chen[2] proposed a fuzzy assignment model. Wang[3] solved a similar model by graph theory techniques. Dominance of fuzzy numbers can be explained by many ranking methods[4,5]. Of these, we use Yager's ranking method[4] which satisfies the properties of compensation, linearity and additivity.

Zadeh[1] introduced fuzzy set in the year 1965.

Definition: The fuzzy set can be mathematically constructed by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set[1,6].

Definition: The fuzzy number $\bar{A}$ is a fuzzy set whose membership function $\mu_{\bar{A}}(x)$ satisfies the following condition[7]

a) $\mu_{\bar{A}}(x)$ is a piecewise continuous
b) $\mu_{\bar{A}}(x)$ is a convex
c) $\mu_{\bar{A}}(x)$ is the normal (i.e) $\mu_{\bar{A}}(x_0) = 1$

Definition: A fuzzy number with membership function in the form

$$\mu_{\bar{A}}(x) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 1 & x = b \\ \frac{d-x}{d-c} & b \le x \le c \\ 0 & otherwise \end{cases}$$

Is called a triangular fuzzy number $\bar{A}$ = (a,b,c)

Definition: A fuzzy number with membership function of the form

$$\mu_{\bar{A}}(x) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 1 & b \le x \le c \\ \frac{d-x}{d-c} & c \le x \le d \\ 0 & otherwise \end{cases}$$

is called a Trapezoidal fuzzy number $\bar{A}$=(a,b,c,d)

Operations on trapezoidal numbers and triangular numbers:

Addition : $(a_1,b_1,c_1,d_1) + (a_2,b_2,c_2,d_2) = (a_1+ a_2, b_1+b_2, c_1+c_2, d_1+d_2)$ $(a_1,b_1,c_1)+(a_2,b_2, c_2,) = (a_1+a_2, b_1+b_2, c_1+c_2)$

Subtraction: $(a_1,b_1,c_1,d_1) - (a_2,b_2,c_2,d_2) = (a_1- d_2, b_1-c_2, c_1-b_2, d_1-a_2)$ $(a_1,b_1,c_1)-(a_2,b_2,c_2,) = (a_1- c_2, b_1-b_2, c_1-a_2)$

Yager's ranking method:

Yager's ranking techniques which satisfies compensation, linearity and additive properties and provides results which are consistent with human intuition.

$$Y(\bar{a}) = \int_0^1 (0.5)\,(a_\alpha^L + a_\alpha^U)\, d\alpha$$ where a_α^L= Lower α - level cut, a_α^U = Upper α- level cut, for any two fuzzy numbers $\tilde{s}$ and $\tilde{\imath}$, we have $\tilde{s} \le \tilde{\imath}$ whenever $Y(\tilde{s}) \le Y(\tilde{\imath})$

Extension of addition of the two fuzzy numbers :

Extension of addition of the two fuzzy numbers (a_1, a_2, a_3, a_4) (–) (b_1, b_2, b_3, b_4) is $(b_1, b_2, b_3, b_4)+(x_1+x_2+x_3+x_4) = (a_1, a_2, a_3, a_4)$

$b_1+x_1= a_1,\ x_1= a_1-b_1$
$b_2+x_2= a_2,\ x_2= a_2-b_2$
$b_3+x_3=a_3,\ x_3= a_3-b_3$
$b_4+x_4=a_4,\ x_4= a_4-b_4$

Fuzzy assignment problem:

It can be stated in the form of n×n cost matrix

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$$\begin{array}{c c} & \begin{array}{cccc} job1 & job2 & . & jobN \end{array} \\ \begin{array}{c} person1 \\ person2 \\ person3 \\ . \\ personN \end{array} & \begin{pmatrix} \overline{C_{11}} & \overline{C_{12}} & . & . & \bar{C}_{1N} \\ \bar{C}_{21} & \bar{C}_{22} & . & . & \bar{C}_{2N} \\ \bar{C}_{31} & \bar{C}_{32} & . & . & \bar{C}_{3N} \\ . & . & . & . & . \\ \bar{C}_{N1} & \bar{C}_{N2} & . & . & \bar{C}_{NN} \end{pmatrix} \end{array}$$

Mathematically it can be stated as

$$\text{Minimize } \bar{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{C_{IJ}}\ x_{ij}$$

subject to $\sum_{i=1}^{n} x_{ij}=1$, $\sum_{j=1}^{n} x_{ij} = 1$, where

$$x_{ij} = \begin{cases} 1 & \text{if } i^{th} \text{ person is assigned to the } j^{th} \text{ job} \\ 0 & otherwise \end{cases}$$

Theorem : Crisps values of subtraction of two fuzzy numbers and extension of addition of two fuzzy numbers are equal.

Proof: Let us consider two fuzzy number (a_1,a_2,a_3,a_4) and (b_1,b_2,b_3,b_4). Their subtraction is given by

$(a_1,a_2,a_3,a_4) - (b_1,b_2,b_3,b_4) = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$

Crisps value of the fuzzy number is

$Y(a_1-b_4,a_2-b_3,a_3-b_2,a_4-b_1) = (a_1+a_2+a_3+a_4) -(b_1+b_2+b_3+b_4)/4$ by using (4)

Extension of addition of these two fuzzy number is

$$(b_1, b_2, b_3, b_4)+(x_1+x_2+x_3+x_4) = (a_1, a_2, a_3, a_4)$$
$$b_1+x_1= a_1,\ x_1= a_1-b_1$$
$$b_2+x_2= a_2,\ x_2= a_2-b_2$$
$$b_3+x_3=a_3,\ x_3= a_3-b_3$$
$$b_4+x_4=a_4,\ x_4= a_4-b_4$$

Therefore the crisps value of $(a_1-b_1, a_2-b_2, a_3-b_3, a_4-b_4)$ is $(a_1+a_2+a_3+a_4)-(b_1+b_2+b_3+b_4)/4$

Both are equal.

Reduction theorem:

It states that in an fuzzy assignment problem if we add (or subtract) a constant fuzzy number to every element of any row or column of the cost matrix $(\overline{C_{ij}})$ then an assignment that minimizes the total cost on one matrix will also minimize the total cost on the other matrix.

Proof: let $x_{ij}= X_{ij}$ minimizes the total cost (ie)$\bar{Z}= \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{C_{ij}}\ x_{ij}$ for every $x_{ij} \geq 0$ and $\sum_{i=1}^{n} x_{ij}= \sum_{j=1}^{n} x_{ij} = 1$ it is to be shown that the assignment $x_{ij}= X_{ij}$ also minimizes the total cost

$$\bar{Z}' = \sum_{i=1}^{n}\sum_{j=1}^{n}(\overline{C_{ij}} - \overline{u_i} - \overline{v_j})\ x_{ij}$$

whereand $\overline{u_i}$ are $\overline{v_j}$ fuzzy number constants subtracted from i^th row and j^th column of cost matrix $(\overline{C_{ij}})$

$$\bar{Z}' = \sum_{i=1}^{n}\sum_{j=1}^{n}(\overline{C_{ij}}\ x_{ij}) - \sum_{i=1}^{n}\overline{u_i}\sum_{j=1}^{n} x_{ij} - \sum\sum_{j=1}^{n}\overline{v_j}\sum_{i=1}^{n} x_{ij}$$
$$\bar{Z}' = Z - \sum_{i=1}^{n}\overline{u_i} - \sum_{j=1}^{n}\overline{v_j} \qquad (\text{since } \sum_{i=1}^{n} x_{ij}= \sum_{j=1}^{n} x_{ij} = 1)$$

This implies $\bar{Z}'$ is minimized and conversely

Theorem 2: If all $\overline{C_{ij}} \geq 0$ and it is possible to find a set $x_{ij}= X_{ij}$ so that $\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{C_{ij}}\, x_{ij} = 0$ then this assignment is optimal.

Proof: Since neither $\overline{C_{ij}}$ is negative the values of $\bar{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{C_{ij}}$ xij cannot be negative. Hence its minimum value is zero which is attained $x_{ij} = x^*_{ij}$. Thus the present solution is an optimal solution.

The above theorem indicates that if one can create a new $\overline{C_{ij}}$ matrix with fuzzy zero entries and these fuzzy zero elements or a subset thereof constitute a feasible solution, then this feasible solution is the optimal solution.

We now propose an algorithm for solving a fuzzy assignment problem based on the Hungarian method.

Step: 1

Subtract the minimum of each row of the effectiveness matrix, from all the fuzzy elements of the respective rows. (Subtract means extension of addition)

Step: 2

Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective column. Thus obtained first modified matrix. (Subtract means extension of addition)

Step: 3

Then draw the minimum no of horizontal and vertical lines to cover all the fuzzy zero elements in the resulting matrix. If these may be two possibilities a) if N= n (number of rows) then an optimal assignment can be made. So make the zero assignment to get the required solution. b) if N< n , then proceed to step 4.

Step: 4

Determine the smallest fuzzy element in the matrix not covered by N lines. Use extension of addition method to all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step: 5

Again repeat the steps 3 and 4 until minimum no of lines become equal to the number of rows or columns (i.e.) N=n

Step: 6

Examine the rows successively until a row-wise exactly single fuzzy zero element is found. Mark this fuzzy zero element by ( ) to make the assignment, then crossover all fuzzy zero elements is lying in the respective column, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined.

Step: 7

Repeat the step 6 successively until one of the following situation arises:

a) if no unmarked fuzzy zero element is left, then the process ends or

b) if there lie more than one unmarked fuzzy zero elements in any column or row, then mark the one of the unmarked fuzzy zero elements arbitrary and a cross in the cells of remaining fuzzy zero elements in its row or column. Repeat the process until no unmarked fuzzy zero element is left in the matrix.

Step: 8

Thus exactly one marked fuzzy zero element in each row and each column of the fuzzy cost matrix is obtained.

The assignment corresponding to these marked fuzzy zero elements gives the optimal assignment.

Numerical Examples:

Example 1 :

Let us consider a fuzzy assignment problem with rows representing 5 persons A,B,C,D,E and columns representing the 5 jobs Job1,Job2,Job3,Job4,Job5 The cost matrix is ($\overline{C_{ij}}$) is given in trapezoidal fuzzy numbers

$$(\overline{C_{ij}}) = \begin{pmatrix} (4,6,7,9) & (3,5,7,9,) & (5,7,10,12) & (3,4,6,9) & (4,5,7,10) \\ (2,3,5,9) & (5,7,9,13) & (4,6,9,12) & (5,6,7,10) & (2,3,5,7) \\ (7,9,10,12) & (6,7,9,10) & (7,9,10,13) & (6,7,10,13) & (7,10,13,14) \\ (4,5,7,9) & (5,7,12,15) & (7,9,13,15) & (2,9,10,13) & (5,7,10,14) \\ (4,10,13,15) & (3,7,9,13) & (2,3,10,14) & (3,7,10,13) & (4,7,10,14) \end{pmatrix}$$

Solution:

Row reduction matrix:

$$= \begin{pmatrix} (1,1,2,0) & (0,1,1,1) & (2,3,3,4) & (0,0,0,0) & (1,1,1,1) \\ (0,0,0,2) & (3,4,4,6) & (2,3,4,5) & (2,3,3,3) & (0,0,0,0) \\ (1,1,2,2) & (0,0,0,0) & (1,1,2,3) & (0,0,1,3) & (1,3,4,4) \\ (0,0,0,0) & (1,2,5,6) & (3,4,6,6) & (-2,-1,3,4) & (1,2,3,5) \\ (1,2,3,7) & (-1,-1,1,4) & (0,0,0,0) & (-1,0,1,4) & (0,0,2,4) \end{pmatrix}$$

Column reduction matrix is the same.

$$= \begin{pmatrix} (1,1,2,0) & (0,1,1,1) & (2,3,3,4) & (0,0,0,0) & (1,1,1,1) \\ (0,0,0,2) & (3,4,4,6) & (2,3,4,5) & (2,3,3,3) & (0,0,0,0) \\ (1,1,2,2) & (0,0,0,0) & (1,1,2,3) & (0,0,1,3) & (1,3,4,4) \\ (0,0,0,0) & (1,2,5,6) & (3,4,6,6) & (-2,-1,3,4) & (1,2,3,5) \\ (1,2,3,7) & (-1,-1,1,4) & (0,0,0,0) & (-1,0,1,4) & (0,0,2,4) \end{pmatrix}$$

The minimum number of lines = number of rows.

$$= \begin{pmatrix} (1,1,2,0) & (0,1,1,1) & (2,3,3,4) & (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) & (1,1,1,1) \\ (0,0,0,2) & (3,4,4,6) & (2,3,4,5) & (2,3,3,3) & (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \\ (1,1,2,2) & (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) & (1,1,2,3) & (0,0,1,3) & (1,3,4,4) \\ (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) & (1,2,5,6) & (3,4,6,6) & (-2,-1,3,4) & (1,2,3,5) \\ (1,2,3,7) & (-1,-1,1,4) & (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) & (-1,0,1,4) & (0,0,2,4) \end{pmatrix}$$

A$\rightarrow$4 , B$\rightarrow$5 C$\rightarrow$2 , D$\rightarrow$1, E$\rightarrow$3 The fuzzy optimal total cost is = (3,4,6,9)+ (2,3,5,7) +(6,7,9,10) + (4,5,7,9) + (2,3,10,14)= (17,22,37,49)

Example: 2

Let us consider a fuzzy assignment problem with rows representing 4 persons A,B,C,D and columns representing the 4 jobs Job1,Job2,Job3,Job4. The cost matrix is ($\overline{C_{ij}}$) is given in triangular fuzzy numbers

$$(\overline{C_{ij}}) = \begin{pmatrix} (5,10,15) & (5,10,20) & (5,15,20) & (5,10,15) \\ (5,10,20) & (5,15,20) & (5,10,15) & (10,15,20) \\ (5,10,20) & (10,15,20) & (10,15,20) & (5,10,15) \\ (10,15,25) & (5,10,15) & (10,20,30) & (10,15,25) \end{pmatrix}$$

Row reduction matrix:

$$= \begin{pmatrix} (0,0,0) & (0,0,5) & (0,5,5) & (0,0,0) \\ (0,0,5) & (0,5,5) & (0,0,0) & (5,5,5) \\ (0,0,5) & (5,5,5) & (5,5,5) & (0,0,0) \\ (5,5,10) & (0,0,0) & (5,10,15) & (5,5,10) \end{pmatrix}$$

Column reduction matrix is the same.

$$= \begin{pmatrix} (0,0,0) & (0,0,5) & (0,5,5) & (0,0,0) \\ (0,0,5) & (0,5,5) & (0,0,0) & (5,5,5) \\ (0,0,5) & (5,5,5) & (5,5,5) & (0,0,0) \\ (5,5,10) & (0,0,0) & (5,10,15) & (5,5,10) \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{0},\mathbf{0},\mathbf{0}) & (0,0,5) & (0,5,5) & (0,0,0) \\ (0,0,5) & (0,5,5) & (\mathbf{0},\mathbf{0},\mathbf{0}) & (5,5,5) \\ (0,0,5) & (5,5,5) & (5,5,5) & (\mathbf{0},\mathbf{0},\mathbf{0}) \\ (5,5,10) & (\mathbf{0},\mathbf{0},\mathbf{0}) & (5,10,15) & (5,5,10) \end{pmatrix}$$

This implies $A \to 1$, $B \to 3$, $C \to 4$, $D \to 2$

The assignment cost is = (5,10,15) + (5,10,15) + (5,10,15) + (5,10,15) = (20,40,60)

Conclusion

A new algorithm is proposed for solving fuzzy assignment problem. This algorithm is effective and easy to understand because of its similarity to Hungarian method.

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