On Weak Concircular Symmetries of ∈-Trans Sasakian manifolds

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Abstract

The object of the present paper is to study weakly concircular symmetric and weakly concircular Ricci symmetric ∈-Trans-Sasakian manifolds and the relationship among the 1-forms A, B and D for locally symmetry in a weakly symmetric and Ricci symmetry of ∈-Trans-Sasakian manifolds. Also the geometric meaning of 1-forms B and D for recurrency. Geometrical meaning for B and D for Ricci recurrency in a weakly Ricci symmetric ∈-Trans-Sasakian manifolds.

1 Introduction

In 2011, Shyamal Kumar[12] Hui has studied the weak concircular Symmetries of trans-Sasakian manifolds. In 1989 Tamassy and Binh[9] have introduced the following definition:

Definition 1.1 A non flat Riemannian manifold (M^n, g) (n>2) is called a weakly symmetric manifold if its curvature tensor R of type (0, 4) satisfies the condition[1,2]

$$\nabla_X R)(Y, Z, U, V) = A(X) R(Y, Z, U, V)$$
$$+ B(Y)(X, Z, U, V)$$

$$+ H(Z) R(Y, X, U, V) + D(U) R(Y, Z, X, V)$$
$$+ E(V) R(Y, Z, U, X) \tag{1.1}$$

for all vector fields X, Y, Z, U, V $\in \chi(M^n); \chi(M)$ being the Lie algebra of smooth vector fields of M, where A,B,H,D and E are 1-forms and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. In 1999 De and Bandyopadhyay[3] studied a and proved that in such a manifold the associated 1-forms B=H and D=E. hence (1.1) reduces to the following:

$$\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V)$$
$$+ B(Y)(X, Z, U, V)$$
$$+ B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V)$$
$$+ D(V)R(Y, Z, U, X) \tag{1.2}$$

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation[11]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined[11] by

$$\tilde{C}(Y, Z, U, V) = R(Y, Z, U, V) - \frac{r}{n(n-1)}$$
$$[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)], \tag{1.3}$$

where r is the scalar curvature of the manifold.

In 2009, Shaikh and Hui[7,8] have introduced the following definition:

Definition 1.2 A Riemannian manifold (M^n, g) (n>2) is called weakly concircular symmetric manifold if its concircular curvature tensor $\tilde{C}$, of type (0, 4) is not identically zero and satisfies the condition

$$(\nabla_X \tilde{C})(Y, Z, U, V) = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V)$$
$$+ H(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V)$$
$$+ E(V)\tilde{C}(Y, Z, U, X) \tag{1.4}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, where A, B, H, D and E are 1- forms an n-dimensional manifold of this kind is denoted by $(W\tilde{C}S)_n$. Also it is shown that[7], in a the associated 1-forms B=H and D=E, and hence

the defining condition (1.4) of a $(W\tilde{C}S)_n$ reduces to the following form:

$$(\nabla_X \tilde{C})(Y, Z, U, V) = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V)$$
$$+ B(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V)$$
$$+ D(V)\tilde{C}(Y, Z, U, X) \tag{1.5}$$

where A, B, D are 1-forms.
In 1993, Tamassy and Binha have introduced the following definition:

Definition 1.3 A Riemannian manifold (M^n, g) (n>2) is called weakly Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X)$$
$$\tag{1.6}$$

where A, B, and D are three non-zero 1-forms, called the associated 1-forms of the manifold, and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(W\tilde{C}S)_n$.

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y, V) = \sum_{i=1}^{n} \tilde{C}(Y, e_i, e_i, V), \tag{1.7}$$

then from (1.3), we get

$$P(Y, V) = S(Y, V) - \frac{r}{n} g(Y, V). \tag{1.8}$$

The tensor P is called the concircular Ricci symmetric tensor[4], which is a symmetric tensor of type (0, 2).

In 2005, De and Ghosh have introduced the following definition:

Definition 1.4 A Riemannian manifold (M^n, g) $(n>2)$ is called weakly concircular Ricci symmetric manifold if its concircular Ricci tensor P of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X P)(Y, Z) = A(X)P(Y, Z) + B(Y)P(X, Z) + D(Z)P(Y, X),$$
$$(1.9)$$

where A, B and D are three 1-forms.

In[13], A. Bejancu and K. L. Duggal introduced the notion of ∈-Sasakian manifolds with indefinite metric. In 1998, Xu Xufeng and Chao Xiaoli proved that every ∈-Sasakian manifold is a hyper surface of an indefinite Kaehlerian manifold and established a necessary and sufficient condition for an odd dimensional Remannian manifold to be an ∈-Sasakian manifolds[14]. In[15], U. C. De and Avijit Sarkar introduced and studied the notion of ∈-Kenmotsu manifolds with indefinite metric giving an example.

Section 2 is devoted to the preliminary results of ∈-Trans-Sasakian manifolds that are needed in the rest of the sections.Recently S. K. Hui[12] studied weak concircular Symmetries of Trans-Sasakian manifolds. However, in section 3 of the paper we have obtained all the 1-forms of a weakly concircular Symmetric ∈-Trans-Sasakian manifolds and hence such a structure exist always. In section 4 we study weakly concircular Ricci Symmetric ∈-Trans-Sasakian manifolds and obtained all the 1-forms of a weakly concircular Ricci Symmetric ∈-Trans-Sasakian manifold and consequently such a structure is always exist. Also it is proved that the sum of the associated 1-forms

of a weakly concircular Ricci Symmetric ∈-Trans-Sasakian manifold is non-vanishing everywhere.

In section 5, the geometrical meaning of the new 1-forms B and D appeared in the definition 1.5 is obtained if a weakly concircular Symmetric ∈-Trans-Sasakian manifolds is locally symmetric, relationship among the 1-forms A, B and D of the definition 1.5 and geometrical meaning of 1-forms B and D for recurrency is also obtained. In section 6, the geometrical meaning of the new 1-forms B and D appeared in the definition 1.9 is obtained if a weakly concircular Symmetric ∈-Trans-Sasakian manifolds is locally Ricci symmetric, relationship among the 1-forms A, B and D of the definition 1.9 and geometrical meaning of 1-forms B and D for recurrency is also obtained. Some of the corollaries in each theorem are given. In this section of the paper wherein we have provided an concrete example for the existence of weakly Concircular Ricci-symmetric ∈-Trans-Sasakian manifolds.

Some of the results of Shamal Kumar Hui[12] shall includes as a special cases of our results for ∈=1.

2. *Preliminaries :*

In this section, we list the basic definitions and known results of ∈-Trans-Sasakian manifolds.

Definition[16] *2.1.* A (2n+1)-dimensional differentiable manifold (M, g) is said to be an ∈-almost contact metric manifold, if it admits a (1, 1) tensor field ϕ, a structure vector field ξ, a 1-form η an indefinite metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \qquad (2.1)$$

$$g(\xi, \xi) = \in, \eta(X) = \in g(X, \xi) \qquad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \in \eta(X)\eta(Y) \quad (2.3)$$

for all vector fields X, Y on M, where $\in$ is 1 or -1 according as ξ is space like or time like and rank ϕ is 2n.

From the above equations, one can deduce that

$$\phi\xi = 0, \eta(\phi X) = 0$$

Definition 2.2 An $\in$-almost contact metric manifold is called an $\in$-Trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \in \eta(Y)X\}$$

$$+ \beta\{g(\phi X, Y)\xi - \in \eta(Y)\phi X\}, (2.4)$$

for any X, Y on M, where ∇ is Live-Civita connection with respect to g.

We note that if $\in=1$, i.e. structure vector field ξ is space like, then an $\in$-Trans – Sasakian manifold is usual trans-Sasakian manifold[5].

A Trans-Sasakian manifold of type (0, 0), (0,β), (α, 0) are the cosympletic, β-Kenmotsu and α-Sasakian manifolds respectively. In particular if $\alpha=1,\beta=0$, and $\alpha=0,\beta=1$, then trans-Sasakian manifold reduces to Sasakian and Kenmotsu manifolds respectively. For $\in$-Trans –Sasakian manifold, we have[17]

$$(\nabla_X \xi) = \in \{-\alpha\phi X + \beta(X - \eta(X)\xi\} \quad (2.5)$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta\{g(X, Y) - \in \eta(X)\eta(Y)\} \qquad (2.6)$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\}$$

$$+ 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\}$$

$$+ \in \{(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y\} \qquad (2.7)$$

$$\eta(R(X, Y)Z) = \in (\alpha^2 - \beta^2)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}$$

$$+ 2 \in \alpha\beta\{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\}$$

$$+ \{(X\beta)g(\phi^2 Y, Z) - (Y\beta)g(\phi^2 X, Z)\}$$

$$+ \{(X\alpha)g(\phi Y, Z) - (Y\alpha)g(\phi X, Z)\} \qquad (2.8)$$

$$S(X, \xi) = \{2n(\alpha^2 - \beta^2) - \in (\xi\beta)\}\eta(X)$$

$$- \in (\phi X)\alpha - \in (2n - 1)(X\beta) \quad (2.9)$$

$$R(\xi, X)\xi = \{\alpha^2 - \beta^2 - \in (\xi\beta)\}\{-X + \eta(X)\xi\}$$

$$- \{2\alpha\beta + \in (\xi\alpha)\}(\phi X) \qquad (2.10)$$

$$S(\xi, \xi) = 2n\{\alpha^2 - \beta^2 - \in (\xi\beta)\} (2.11)$$

$$2\alpha\beta + \in (\xi\alpha) = 0 \qquad (2.12)$$

where R is the curvature tensor of type (1, 3) of the manifold and S is Ricci tensor.

3. Weakly Concircular symmetric $\in$-Trans-Sasakian manifolds:

Definition 3.1 A $\in$-Trans-Sasakian manifolds (M^{2n+1},g) (n>1) is said to be weakly concircular symmetric if its concircular curvature tensor $\tilde{C}$ of type (0, 4) satisfies (1.5).

Setting Y=V=e_i in (1.5) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$(\nabla_X S)(Z, U) - \frac{dr(X)}{n} g(Z, U)$$

$$= A(X)[S(Z, U) - \frac{r}{n} g(Z, U)] + B(Z)[S(X, U) - \frac{r}{n} g(X, U)]$$

$$+ D(U)[S(X, Z) - \frac{r}{n} g(X, Z)] + B(R(X, Z)U) + D(R(X, U)Z)$$

$$- \frac{r}{n(n-1)}[\{B(X) + D(X)\}g(Z, U) - B(Z)g(X, U) - D(U)g(Z, X)] \quad (3.1)$$

Putting $X=Z=U=\xi$ in (3.1) and then using (2.7) and (2.11), we obtain

$$A(\xi) + B(\xi) + D(\xi) = \frac{2n^2\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \in \xi(\xi\beta)\} - \in dr(\xi)}{2n^2\{\alpha^2 - \in (\xi\beta) - \beta^2\} - \in r} \quad (3.2)$$

Hence we can state the following:

Lemma 3.1 In a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g) $(n>1)$, the relation (3.2) holds.

Next, substituting X and Z by ξ in (3.1) and then using (2.7) and (2.12) we obtain

$$(\nabla_\xi S)(\xi, U) - \frac{dr(\xi)}{\in n} \eta(U)$$

$$= [A(\xi) + B(\xi)][S(U, \xi) - \frac{r}{\in n} \eta(U)]$$

$$+ D(U)[(2n-1)\{\alpha^2 - \beta^2 - \in (\xi\beta)\} - \frac{(n-2)r}{\in n(n-1)}]$$

$$+ [\alpha^2 - \in (\xi\beta) - \beta^2 - \frac{r}{\in n(n-1)}]\eta(U)D(\xi) .$$

$$(3.3)$$

From (2.9), we have

$$(\nabla_\xi S)(\xi, U) = \nabla_\xi S(\xi, U) - S(\nabla_\xi \xi, U) - S(\xi, \nabla_\xi U)$$

$$= \nabla_\xi S(\xi, U) - S(\xi, \nabla_\xi U)$$

$$= [2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - \in \xi(\xi\beta)]\eta(U)$$

$$-(2n-1) \in U(\xi\beta) - \in (\phi U)(\xi\alpha) ,(3.4)$$

where (2.9) has been used. In view of (3.3) and (3.4) we obtain from (3.2) that

$$D(U) = \frac{[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - \in \xi(\xi\beta) - \frac{dr(\xi)}{\in n}]\eta(U)}{(2n-1)[\alpha^2 - \beta^2 - \in (\xi\beta)] - \frac{(n-2)r}{\in n(n-1)}}$$

$$- \frac{(2n-1) \in U(\xi\beta) + \in (\phi U)(\xi\alpha)}{(2n-1)[\alpha^2 - \beta^2 - \in (\xi\beta)] - \frac{(n-2)r}{\in n(n-1)}}$$

$$+ D(\xi)[\frac{(2n-1)(\alpha^2 - \beta^2)\eta(U) - \in (2n-1)(U\beta) - \in (\phi U)\alpha - \frac{(n-2)}{\in n(n-1)}r\eta(U)}{(2n-1)[\alpha^2 - \beta^2 - \in (\xi\beta)] - \frac{(n-2)r}{\in n(n-1)}}]$$

$$-\frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\in\xi(\xi\beta)\}-\dfrac{dr(\xi)}{n}}{[2n\{\alpha^2-\beta^2-\in(\xi\beta)\}-\dfrac{\in r}{n}][(2n-1)\{\alpha^2-\beta^2-\in(\xi\beta)\}-\dfrac{(n-2)r}{\in n(n-1)}]}$$

$$[\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\frac{r}{\in n}\}\eta(U)-\in(2n-1)(U\beta)-\in(\phi U)\alpha]. \tag{3.5}$$

Next, setting X=U=ξ in (3.1) and proceeding in a similar manner as above, we get

$$B(Z)=\frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-\in\xi(\xi\beta)-\dfrac{dr(\xi)}{\in n}]\eta(Z)}{(2n-1)[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{(n-2)r}{\in n(n-1)}}$$

$$-\frac{(2n-1)\in Z(\xi\beta)+\in(\phi Z)(\xi\alpha)}{(2n-1)[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{(n-2)r}{\in n(n-1)}}$$

$$+B(\xi)[\frac{(2n-1)(\alpha^2-\beta^2)\eta(Z)-\in(2n-1)(Z\beta)-\in(\phi Z)\alpha-\dfrac{(n-2)}{\in n(n-1)}r\eta(Z)}{(2n-1)[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{(n-2)r}{\in n(n-1)}}]$$

$$-\frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\in\xi(\xi\beta)\}-\dfrac{dr(\xi)}{n}}{[2n\{\alpha^2-\beta^2-\in(\xi\beta)\}-\dfrac{\in r}{n}][(2n-1)\{\alpha^2-\beta^2-\in(\xi\beta)\}-\dfrac{(n-2)r}{\in n(n-1)}]}$$

$$[\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\frac{r}{\in n}\}\eta(Z)-\in(2n-1)(Z\beta)-\in(\phi Z)\alpha] \tag{3.6}$$

for any vector field Z. Hence we can state the following:

Theorem 3.2 In a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1), the associated 1-forms D and B are given by (3.5) and (3.6) respectively.
Again setting Z=U=ξ, in (3.1), we get

$$(\nabla_X S)(\xi,\xi)-\frac{\in dr(X)}{n}$$

$$= A(X)[S(\xi,\xi) - \frac{\in r}{n}] + [B(\xi) + D(\xi)][S(X,\xi) - \frac{n-2}{\in n(n-1)} r\eta(X)]$$

$$+ B(R(X,\xi)\xi) + D(R(X,\xi)\xi) - \frac{\in r}{n(n-1)}[B(X) + D(X)]$$

$$= [2n\{\alpha^2 - \beta^2 - \in(\xi\beta)\} - \frac{\in r}{n}]A(X)$$

$$+ [B(\xi) + D(\xi)][S(X,\xi) - \{\frac{n-2}{\in n(n-1)}r + \alpha^2 - \in(\xi\beta) - \beta^2\}\eta(X)]$$

$$+ [B(X) + D(X)][\alpha^2 - \in(\xi\beta) - \beta^2 - \frac{\in r}{n(n-1)}]. \tag{3.7}$$

Now we have

$$(\nabla_X S)(\xi,\xi) = \nabla_X S(\xi,\xi) - 2S(\nabla_X \xi,\xi),$$

which yields by using (2.5) and (2.11) that

$$(\nabla_X S)(\xi,\xi) = 2n\{2\alpha(X\alpha) - 2\beta(X\beta) - \in X(\xi\beta)\}$$
$$+ 2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)(\phi X)\beta]$$
$$+ 2\beta[(\phi X)\alpha + (2n-1)\{X\beta - (\xi\beta)\eta(X)\}] \tag{3.8}$$

In view of (3.5), (3.6) and (3.7), (3.8) yields for any vector field,

$$A(X) + B(X) + D(X)$$

$$= \frac{2n[2\alpha(X\alpha) - 2\beta(X\beta) - \in X(\xi\beta)]}{[2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}]}$$

$$+ \frac{2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)(\phi X)\beta]}{[2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}]}$$

$$+ \frac{2\beta[(\phi X)\alpha + (2n-1)\{X\beta - (\xi\beta)\eta(X)\}] - \frac{\in dr(X)}{n}}{[2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}]}$$

$$+ \frac{2[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \in\xi(\xi\beta) - \frac{dr(\xi)}{\in n}]\eta(X)}{[2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}]}$$

$$- \frac{[(2n-1) \in X(\xi\beta) + \in(\phi X)\xi\alpha]}{[n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{2n}]}$$

$$- \frac{2[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \in\xi(\xi\beta)\} - \frac{\in dr(\xi)}{n}]}{[2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}]^2}$$

$$[\{2n(\alpha^2 - \beta^2) - \in(\xi\beta) - \frac{r}{\in n}\}\eta(X)$$

$$- \in(2n-1)(X\beta) - \in(\phi X)\alpha] \tag{3.10}$$

This leads to the following:

Theorem 3.3 In a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g) $(n>1)$, the sum of the associated 1-forms is given by (3.10).

In particular, if $\phi(\text{grad}\alpha) = \text{grad}\beta$, then $(\xi\beta) = 0$ and hence the relation (3.10) reduces to the following form

A(X) +B(X) +D(X)

$$= \frac{2n[2\alpha(X\alpha) - 2\beta(X\beta)]}{2n(\alpha^2 - \beta^2) - \frac{\in r}{n}} + \frac{2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)(\phi X)\beta]}{2n(\alpha^2 - \beta^2) - \frac{\in r}{n}}$$

$$+ \frac{2\beta[(\phi X)\alpha + (2n-1)(X\beta)] - \frac{\in dr(X)}{n} + 2[4n\alpha(\xi\alpha) - \frac{dr(\xi)}{\in n}]\eta(X) - 2 \in (\phi X(\xi\alpha))}{2n(\alpha^2 - \beta^2) - \frac{\in r}{n}}$$

$$- \frac{2[4n\alpha(\xi\alpha) - \frac{\in dr(\xi)}{n}]}{2n(\alpha^2 - \beta^2) - \frac{\in r}{n}}[\{2n(\alpha^2 - \beta^2) - \frac{r}{\in n}\}\eta(X) - \in (2n-1)(X\beta) - \in (\phi X)\alpha] \quad (3.11)$$

for any vector field X. This leads to the following:

Corollary 3.4 If a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1), satisfies the condition $\phi(grad\alpha) = grad\beta$, then the sum of the associated 1-forms is given by (3.11).

If $\beta=0$ and $\alpha=1$, then (3.10) yields A(X)+B(X)+D(X)=0 for all X and hence we can state the following:

Corollary 3.5 There is no weakly concircular symmetric $\in$-Sasakian manifold (M^{2n+1},g) (n>1), unless the sum of the 1-forms is everywhere zero.

Corollary 3.6 If an $\in$-α-Sasakian manifold is weakly concircular symmetric, then the sum of the 1-forms *i.e.*, A+B+D is given by

A(X)+B(X)+D(X)

$$= \frac{2\alpha[(2n-1)(X\alpha) - \eta(X)(\xi\alpha)] - 2(\phi X(\xi\alpha))}{2n\alpha^2 - \frac{\in r}{n}}$$

$$+ \frac{2[4\alpha n(\xi\alpha) - \frac{dr(\xi)}{\in n}]\eta(X)}{2n\alpha^2 - \frac{\in r}{n}}$$

$$- \frac{2[4n\alpha(\xi\alpha) - \frac{\in dr(\xi)}{n}][\{2n\alpha^2 - \frac{r}{\in n}\}\eta(X) - \in (\phi X)\alpha]}{(2n\alpha^2 - \frac{\in r}{n})^2}.$$

$$(3.12)$$

Again, if $\alpha=0$ and $\beta=1$, then (3.10) yields A(X)+B(X)+D(X)=0 for all X. This leads to the following:

Corollary 3.6 There is no weakly concircular symmetric $\in$-Kenmotsu manifold (M^{2n+1},g) (n>1), unless the sum of the 1-forms is everywhere zero.

Corollary 3.7 If a $\in$-β-Kenmotsu manifold is weakly concircular symmetric, then the sum of the 1-forms *i.e.*, A+B+D is given by

$A(X) + B(X) + D(X)$ (3.13)

$$= \frac{2n\{2\beta(X\beta) + \in (X(\xi\beta))\} - 2(2n-1)\beta\{(X\beta) - (\xi\beta)\eta(X)\} + \frac{\in dr(X)}{n}}{2n\{\in (\xi\beta) + \beta^2\} + \frac{\in r}{n}}$$

$$+ \frac{2[\{4n\beta(\xi\beta) + \xi(\xi\beta) + \frac{dr(\xi)}{\in n}\}\eta(X) + (2n-1) \in X(\xi\beta)]}{2n\{\in (\xi\beta) + \beta^2\} + \frac{\in r}{n}}$$

$$- \frac{2[2n\{2\beta(\xi\beta) + \in \xi(\xi\beta)\} + \frac{\in dr(\xi)}{n}][\{2n\beta^2 + \in (\xi\beta) + \frac{r}{\in n}\}\eta(X) + \in (2n-1)(X\beta)]}{[2n\{\in (\xi\beta) + \beta^2\} + \frac{\in r}{n}]^2}.$$

4. Weakly Concircular Ricci symmetric $\in$- Trans-Sasakian manifolds :

Definition 4.1. A $\in$- Trans-Sasakian manifolds (M^{2n+1}, g) $(n>1)$, is said to be weakly concircular Ricci symmetric if its concircular Ricci tensor P of type $(0,2)$ satisfies (1.9). In view of (1.8), (1.9) yields

$$(\nabla_X S)(Y,Z) - \frac{dr(X)}{n} g(Y,Z) = A(X)[S(Y,Z) - \frac{r}{n} g(Y,Z)]$$

$$+ B(Y)[S(Y,Z) - \frac{r}{n} g(Y,Z)]$$

$$+ D(Z)[S(X,Y) - \frac{r}{n} g(X,Y)] \quad (4.1)$$

Setting $X=Y=Z=\xi$ in (4.1), we get the relation (3.2) and hence we can state the following:

Theorem 4.1 In a weakly concircular Ricci symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g) $(n>1)$, the relation (3.2) holds. Next, substituting X and Y by ξ in (4.1), we obtain

$$(\nabla_\xi S)(\xi,Z) - \frac{dr(\xi)}{\in n}\eta(Z) = [A(\xi) + B(\xi)][S(\xi,Z) - \frac{r}{\in n}\eta(Z)]$$

$$+ D(Z)[S(\xi,\xi) - \frac{r \in}{n}]. \quad (4.2)$$

Using (3.2) and (3.4) in (4.2), we get

$$D(Z) = \frac{[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - \in \xi(\xi\beta) - \frac{dr(\xi)}{\in n}]\eta(Z)}{2n[\alpha^2 - \beta^2 - \in (\xi\beta)] - \frac{r \in}{n}}$$

$$-\frac{[(2n-1)\in Z(\xi\beta)+\in(\phi Z)(\xi\alpha)]}{2n[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{r\in}{n}}$$

$$+D(\xi)[\frac{\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\dfrac{r}{\in n}\}\eta(Z)-\in(\phi Z)\alpha-\in(2n-1)(Z\beta)}{2n[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{r\in}{n}}]$$

$$-\frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\in\xi(\xi\beta)\}-\dfrac{\in dr(\xi)}{n}}{[2n(\alpha^2-\beta^2-\in(\xi\beta)-\dfrac{r\in}{n}]^2}$$

$$[\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\dfrac{r}{\in n}\}\eta(Z)-\in(2n-1)(Z\beta)-\in(\phi Z)\alpha] \qquad (4.3)$$

for all Z.

Again putting X=Z=ξ in (4.1) and proceeding in a similar manner as above we get

$$B(Y)=\frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-\in\xi(\xi\beta)-\dfrac{dr(\xi)}{\in n}]\eta(Y)}{2n[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{r\in}{n}}$$

$$-\frac{[(2n-1)\in Y(\xi\beta)+\in(\phi Y)(\xi\alpha)]}{2n[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{r\in}{n}}$$

$$+B(\xi)[\frac{\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\dfrac{r}{\in n}\}\eta(Y)-\in(\phi Y)\alpha-\in(2n-1)(Y\beta)}{2n[\alpha^2-\beta^2-\in(\xi\beta)]-\dfrac{r\in}{n}}]$$

$$-\frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\in\xi(\xi\beta)\}-\dfrac{\in dr(\xi)}{n}}{[2n(\alpha^2-\beta^2-\in(\xi\beta)-\dfrac{r\in}{n}]^2}$$

$$[\{2n(\alpha^2 - \beta^2) - \in (\xi\beta) - \frac{r}{\in n}\}\eta(Y) - \in (2n-1)(Y\beta) - \in (\phi Y)\alpha] \tag{4.4}$$

for all Y.

Again setting Y=Z=ξ in (4.1) and using (2.9) and (2.11), we get

$$(\nabla_X S)(\xi, \xi) - \frac{\in dr(X)}{n} = [2n(\alpha^2 - \in (\xi\beta) - \beta^2) - \frac{\in r}{n}]A(X)$$

$$+[B(\xi) + D(\xi)][\{2n(\alpha^2 - \beta^2) - \in (\xi\beta) - \frac{r}{\in n}\}\eta(X)$$

$$- \in (\phi X)\alpha - \in (2n-1)(X\beta)]. \tag{4.5}$$

Using (3.2) and (3.8) in (4.5), we get

$$A(X) = \frac{2n\{2\alpha(X\alpha) - 2\beta(X\beta) - \in X(\xi\beta)\}}{2n(\alpha^2 - \in (\xi\beta) - \beta^2) - \frac{\in r}{n}}$$

$$+\frac{2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)(\phi X)\beta]}{2n\{\alpha^2 - \in (\xi\beta) - \beta^2\} - \frac{\in r}{n}}$$

$$+\frac{2\beta[(\phi X)\alpha + (2n-1)\{X\beta - (\xi\beta)\eta(X)\}] - \frac{\in dr(X)}{n}}{2n(\alpha^2 - \in (\xi\beta) - \beta^2) - \frac{\in r}{n}}$$

$$+A(\xi)[\frac{\{2n(\alpha^2 - \beta^2) - \in (\xi\beta) - \frac{r}{\in n}\}\eta(X) - \in (\phi X)\alpha - \in (2n-1)(X\beta)}{2n(\alpha^2 - \in (\xi\beta) - \beta^2) - \frac{\in r}{n}}]$$

$$-\frac{2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \in \xi(\xi\beta)\} - \frac{\in dr(\xi)}{n}}{[2n(\alpha^2 - \in (\xi\beta) - \beta^2) - \frac{\in r}{n}]^2}$$

$$[\{2n(\alpha^2 - \beta^2) - \in (\xi\beta) - \frac{r}{\in n}\}\eta(X) - \in (\phi X)\alpha - \in (2n-1)(X\beta)] \tag{4.6}$$

for all X. This leads to the following:

Theorem 4.2 In a weakly concircular Ricci symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g) (n>1), the associated 1-forms D, B and A are given by (4.3), (4.4) and (4.6) respectively.

Adding (4.3), (4.4) and (4.6) and using (3.2) we get the relation (3.10) and hence we can state the following:

Theorem 4.3 In a weakly concircular Ricci symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g) (n>1), the sum of the associated 1-forms is given by (3.10).

Corollary 4.4 If a weakly concircular Ricci symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g) (n>1), satisfies the condition $\phi(\mathrm{grad}\alpha)=\mathrm{grad}\beta$, then the sum of the associated 1-forms is given by (3.11).

If $\beta=0$ and $\alpha=1$, then (3.10) yields $A(X)+B(X)+D(X)=0$ for all X and hence we can state the following:

Corollary 4.5 There is no weakly concircular Ricci symmetric $\in$-Sasakian manifold (M^{2n+1}, g) (n>1), unless the sum of the 1-forms is everywhere zero.

Corollary 4.6 If an $\in$-α-Sasakian manifold is weakly concircular Ricci symmetric, then the sum of the 1-forms i.e., $A+B+D$ is given by (3.12).

Again, if $\alpha=0$ and $\beta=1$, then (3.10) yields $A(X)+B(X)+D(X)=0$ for all X. This leads to the following:

Corollary 4.6 There is no weakly concircular Ricci symmetric $\in$-Kenmotsu manifold (M^{2n+1}, g) (n>1), unless the sum of the 1-forms is everywhere zero.

Corollary 4.7 If a $\in$-β-Kenmotsu manifold is weakly concircular Ricci symmetric, then the sum of the 1-forms i.e., $A+B+D$ is given by (3.13).

Special cases of Weakly Symmetric Concircular $\in$-Trans-Sasakian manifolds
5. *Locally Symmetric and Recurrent spaces:*

In this section, we consider the locally symmetric spaces in the sense of E. Carton.

Definition 5.1 A weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1}, g) (n>1), is said to be Locally symmetric if its concircular curvature tensor $\tilde{C}$, of type (0, 4) is not identically zero and satisfies the condition

$$(\nabla_X \tilde{C})(Y, Z, U, V) = 0.$$

Suppose weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1}, g) (n>1), is locally symmetric. Then $\nabla\tilde{C} = 0$ so that from (1.5), we have

$$0 = A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V)$$
$$+ B(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V)$$
$$+ D(V)\tilde{C}(Y, Z, U, X) , \qquad (5.1)$$

where X, Y, Z, U and V are vector fields on M. Let $\{e_i : i = 1, 2, ..., 2n + 1\}$ be the orthonormal basis of the tangent space T_pM at any point p of the manifold. Then setting $Y=V=e_i$ in (5.1) and taking summation over i, $1{\leq}i{\leq}2n+1$, we get

$$0 = A(X)[S(Z,U) - \frac{r}{n}g(Z,U)] + B(Z)[S(X,U) - \frac{r}{n}g(X,U)]$$

$$+ D(U)[S(X,Z) - \frac{r}{n}g(X,Z)] + B(R(X,Z)U) + D(R(X,U)Z)$$

$$- \frac{r}{n(n-1)}[\{B(X) + D(X)\}g(Z,U) - B(Z)g(X,U) - D(U)g(Z,X)] \qquad (5.2)$$

Putting X=Z=U=ξ in (5.2) and then using (2.7) and (2.11), we obtain

$$A(\xi) + B(\xi) + D(\xi) = 0 \qquad (5.3)$$

Hence we can state the following:

Lemma 5.1 In a weakly concircular symmetric ∈-Trans-Sasakian manifold (M^{2n+1}, g) (n>1), is locally symmetric the relation (5.3) holds.

Next, substituting X and Z by ξ in (5.2) and

then using (2.7) and (2.12) we obtain

$$0 = [A(\xi) + B(\xi)][S(U,\xi) - \frac{r}{\in n}\eta(U)]$$

$$+ D(U)[(2n-1)\{\alpha^2 - \beta^2 - \in(\xi\beta)\} - \frac{(n-2)r}{\in n(n-1)}]$$

$$+ [\alpha^2 - \in(\xi\beta) - \beta^2 - \frac{r}{\in n(n-1)}]\eta(U)D(\xi). \qquad (5.4)$$

Further using (5.3) in (5.4) we get

$$D(U) = D(\xi)[\frac{(2n-1)(\alpha^2 - \beta^2)\eta(U) - \in(2n-1)(U\beta) - \in(\phi U)\alpha - \frac{(n-2)}{\in n(n-1)}r\eta(U)}{(2n-1)[\alpha^2 - \beta^2 - \in(\xi\beta)] - \frac{(n-2)r}{\in n(n-1)}}] \qquad (5.5)$$

for any vector field U.

Similarly putting X=U=ξ in (5.2) and proceeding in a similar manner as above, we get

$$B(Z) = B(\xi)[\frac{(2n-1)(\alpha^2 - \beta^2)\eta(Z) - \in(2n-1)(Z\beta) - \in(\phi Z)\alpha - \frac{(n-2)}{\in n(n-1)}r\eta(Z)}{(2n-1)[\alpha^2 - \beta^2 - \in(\xi\beta)] - \frac{(n-2)r}{\in n(n-1)}}] \qquad (5.6)$$

for any vector field Z. Hence we can state the following:

Theorem 5.2 If a weakly concircular symmetric ∈-Trans-Sasakian manifold (M^{2+1}, g) (n>1), is locally symmetric, then the associated 1-forms D and B are given by (5.5) and (5.6) respectively.

Now from (5.5) and (5.6), forming B(X)-D(X), we have

$$B(X) - D(X) = \frac{(\alpha^2 - \in (\xi\beta) - \beta^2)\eta(X)\{D(\xi) - B(\xi)\}}{(2n-1)[\alpha^2 - \in (\xi\beta) - \beta^2] - \dfrac{(n-2)r}{\in n(n-1)}} \qquad (5.7)$$

Nest replacing X by $\phi(X)$ in (5.7), we get

$$B(\phi(X)) - D(\phi(X)) = 0$$

Hence we state

Theorem 5.3 If a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1},g) $(n>1)$, is locally symmetric, then

$$B \circ \phi - D \circ \phi = 0$$

Theorem 5.4 If a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1},g) $(n>1)$, with B (ξ) =D (ξ) is locally symmetric, then the associated 1-forms B and D are in the same directions.

Proof. Follows from (5.7).

Corollary 5.5 If a weakly concircular symmetric $\in$-Sasakian manifold (M^{2n+1},g) $(n>1)$, with B (ξ) =D (ξ) is locally symmetric, then the associated 1-forms B and D are in the same directions.

Again setting Z=U=ξ, in (5.2), we get

$$0 = A(X)[S(\xi,\xi) - \frac{\in r}{n}] + [B(\xi) + D(\xi)][S(X,\xi) - \frac{n-2}{\in n(n-1)}r\eta(X)]$$

$$+ B(R(X,\xi)\xi) + D(R(X,\xi)\xi) - \frac{\in r}{n(n-1)}[B(X) + D(X)]$$

$$= [2n\{\alpha^2 - \beta^2 - \in (\xi\beta)\} - \frac{\in r}{n}]A(X)$$

$$+ [B(\xi) + D(\xi)][S(X,\xi) - \{\frac{n-2}{\in n(n-1)}r + \alpha^2 - \in (\xi\beta) - \beta^2\}\eta(X)]$$

$$+ [B(X) + D(X)][\alpha^2 - \in (\xi\beta) - \beta^2 - \frac{\in r}{n(n-1)}]. \qquad (5.8)$$

Adding (5.5), (5.6) taking U=Z=X and then using in (5.8), after simplification, we get

$$A(X) + B(X) + D(X) = 0 \qquad (5.9)$$

for any vector field X on M and $[2n(\alpha^2 - \in (\xi\beta) - \beta^2) - \frac{\in r}{n}] \neq 0$. Further (5.9) can be written as A+B+D=0. Hence we can state

Theorem 5.6 If a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1}, g)

(n>1), is locally symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Corollary 5.7 If a weakly concircular symmetric $\in$-Sasakian manifold (M^{2n+1},g) (n>1), is locally symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Corollary 5.8 If a weakly concircular symmetric $\in$-Kenmotsu manifold (M^{2n+1},g) (n>1), is locally symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Definition 5.10 A weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1), is is said to be recurrent if its concircular curvature tensor $\tilde{C}$, of type (0,4) is not identically zero and satisfies the condition

$$(\nabla_X \tilde{C})(Y,Z,U,V) = A(X)\tilde{C}(Y,Z,U,V) \quad (5.10)$$

where A is associated 1-form and X, Y, Z, U, V are any vector field on M.

Suppose M is Recurrent, then from (1.5) and (5.10), we have

$$0 = B(Y)\tilde{C}(X,Z,U,V) + B(Z)\tilde{C}(Y,X,U,V)$$
$$+ D(U)\tilde{C}(Y,Z,X,V) + D(V)\tilde{C}(Y,Z,U,X)$$
$$(5.11)$$

Setting Y=V=e_i in (5.11), and taking summation over i, $1 \le i \le 2n+1$, we get

$$0 = B(Z)[S(X,U) - \frac{r}{n}g(X,U)]$$
$$+ D(U)[S(X,Z) - \frac{r}{n}g(X,Z)] + B(R(X,Z)U)$$
$$+ D(R(X,U)Z)$$

$$-\frac{r}{n(n-1)}[\{B(X)+D(X)\}g(Z,U)$$
$$-B(Z)g(X,U)-D(U)g(Z,X)] \quad (5.12)$$

Putting X=Z=U=ξ in (5.12) and then using (2.7) and (2.11), we obtain

$$B(\xi) + D(\xi) = 0 \quad\quad\quad (5.13)$$

Now setting $X = Z = \xi$ in (5.12) and then using (2.7) and (2.12) we obtain

$$0 = [B(\xi)][S(U,\xi) - \frac{r}{\in n}\eta(U)]$$
$$+ D(U)[(2n-1)\{\alpha^2 - \beta^2 - \in(\xi\beta)\} - \frac{(n-2)r}{\in n(n-1)}]$$
$$+ [\alpha^2 - \in(\xi\beta) - \beta^2 - \frac{r}{\in n(n-1)}]\eta(U)D(\xi).$$
$$(5.14)$$

By virtue of (5.13), and solving for D (U), we get same equation (5.5). Similarly for (5.6) and (5.7). Thus we obtain the same conclusions as in the Theorem 5.3 and 5.4 and the corollary there under if the 'Locally symmetric' is replaced by 'Recurrent' in the statements.

Special cases of Weakly Ricci Symmetric Concircular $\in$-Trans-Sasakian manifolds
6. Locally Ricci Symmetric and Ricci Recurrent Spaces :

Definition 6.1 A weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1), is said to be Locally Ricci symmetric if its concircular curvature tensor P, of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = 0.$$

Suppose M is locally Ricci symmetric. Then $\nabla P=0$. Therefore from (1.9), we have

$$0 = A(X)P(Y,Z) + B(Y)P(X,Z) + D(Z)P(Y,X)$$
$$(6.1)$$

where A, B and D are three 1-forms. In view of (1.8), (1.9) yields

$$0 = A(X)[S(Y,Z) - \frac{r}{n}g(Y,Z)]$$

$$+B(Y)[S(Y,Z) - \frac{r}{n}g(Y,Z)]$$

$$+D(Z)[S(X,Y) - \frac{r}{n}g(X,Y)] \quad (6.2)$$

Setting X=Y=Z=ξ in (6.2), we get

$$A(\xi) + B(\xi) + D(\xi) = 0 \quad (6.3)$$

Next, substituting X and Y by ξ in (6.2), we obtain

$$0 = [A(\xi) + B(\xi)][S(\xi,Z) - \frac{r}{\in n}\eta(Z)]$$

$$+D(Z)[S(\xi,\xi) - \frac{r \in}{n}] . \quad (6.4)$$

Using (6.3) in (6.4), we get

$$D(Z) = D(\xi)[\frac{\{2n(\alpha^2 - \beta^2) - \in(\xi\beta) - \frac{r}{\in n}\}\eta(Z) - \in(\phi Z)\alpha - \in(2n-1)(Z\beta)}{2n[\alpha^2 - \beta^2 - \in(\xi\beta)] - \frac{r \in}{n}}] \quad (6.5)$$

for any Z.
Similarly, we have

$$B(Y) = B(\xi)[\frac{\{2n(\alpha^2 - \beta^2) - \in(\xi\beta) - \frac{r}{\in n}\}\eta(Y) - \in(\phi Y)\alpha - \in(2n-1)(Y\beta)}{2n[\alpha^2 - \beta^2 - \in(\xi\beta)] - \frac{r \in}{n}}] \quad (6.6)$$

and

$$A(X) = A(\xi)[\frac{\{2n(\alpha^2 - \beta^2) - \in(\xi\beta) - \frac{r}{\in n}\}\eta(X) - \in(\phi X)\alpha - \in(2n-1)(X\beta)}{2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}}] \quad (6.7)$$

Now taking Y=Z=X in(6.5) and (6.6) and then adding (6.5), (6.6) and (6.7) using (6.3) after simplification, we get

$$A(X) + B(X) + D(X) = 0 \quad (6.8)$$

for any vector field X on M and $[2n(\alpha^2 - \in(\xi\beta) - \beta^2) - \frac{\in r}{n}] \neq 0$. Further (6.8) can be written as A+B+D=0. Hence we can state

Theorem 6.2 If a weakly concircular symmetric $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1), is locally Ricci symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Corollary 6.3 If a weakly concircular

symmetric ∈-Sasakian manifold (M^{2n+1},g) $(n>1)$, is locally Ricci symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Corollary 6.4 If a weakly concircular symmetric ∈-Kenmotsu manifold (M^{2n+1},g) $(n>1)$, is locally Ricci symmetric, then the sum of the associated 1-forms A, B and D is zero everywhere.

Definition 6.5 A weakly concircular symmetric ∈-Trans-Sasakian manifold (M^{2n+1},g) $(n>1)$, is is said to be Ricci recurrent if its concircular curvature tensor P, of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = A(X)P(Y,Z).$$

Suppose M is locally Ricci symmetric. Then from (1.9), we have

$$0 = B(Y)P(X,Z) + D(Z)P(Y,X) \qquad (6.9)$$

where A, B and D are three 1-forms.
In view of (1.8), (1.9) yields

$$0 = B(Y)[S(Y,Z) - \frac{r}{n}g(Y,Z)]$$

$$+D(Z)[S(X,Y) - \frac{r}{n}g(X,Y)] \quad (6.10)$$

Setting X=Y=Z=ξ in (6.10), we get

$$B(\xi) + D(\xi) = 0 \qquad (6.11)$$

Next, substituting X and Y by ξ in (6.10), we obtain

$$0 = [B(\xi)][S(\xi,Z) - \frac{r}{\in n}\eta(Z)]$$

$$+D(Z)[S(\xi,\xi) - \frac{r\in}{n}]. \qquad (6.12)$$

Using (6.11) in (6.12), and solving for D (Z), we get same equation (6.5). Similarly for (6.6).

Next adding (6.5) and (6.6) taking Y=Z=X and using (6.11) we get

$$B(X)+D(X)=0.$$

Hence we state

Theorem 6.6 If a weakly concircular symmetric ∈-Trans-Sasakian manifold (M^{2n+1},g) $(n>1)$, is locally Ricci recurrent, then the associated 1-forms B and D are in the opposite directions.

Corollary 6.7 If a weakly concircular symmetric ∈-Sasakian manifold (M^{2n+1},g) $(n>1)$, is locally Ricci recurrent, then the associated 1-forms B and D are in the opposite directions.

Corollary 6.8 If a weakly concircular symmetric ∈-Kenmotsu manifold (M^{2n+1},g) $(n>1)$, is locally Ricci recurrent, then the associated 1-forms B and D are in the opposite directions.

Example: Let us consider a 3-dimensional manifold M={(x, y, z)∈ : z≠0}, where (x, y, z) are the standard coordinates in R^3.

Let $e_1 = e^z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), e_2 = e^z\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z},$

which are linearly independent vector fields at each point of M. define a semi-Riemannian metric g on M as

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, g(e_1,e_1)$$

$$= g(e_2,e_2) = g(e_3,e_3) = \in,$$

where ∈=±1.
Let η be the 1-form defined by $\eta(Z) = \in g(Z,e_3)$,

for any $Z \in \Gamma(TM)$ and ϕ be the tensor field of type (1, 1) defined by $\phi e_1 = e_2, \phi e_2 = -e_1,$ $\phi e_3 = 0$. Then by applying linearity of ϕ and g, we have

$$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3, g(\phi Z, \phi U)$$
$$= g(Z, U) - \in \eta(Z)\eta(U),$$

for any $Z, U \in \Gamma(TM)$. Hence for $e_3 = \xi$, $(\phi,$

$\xi, \eta, g, \in)$ defines an $\in$- almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of type (1, 3), then we have

$$[e_1, e_2] = \in (ye^z e_2 - e^{2z}e_3), [e_1, e_3]$$
$$= -\in e_1, [e_2, e_3] = -\in e_2.$$

By using Koszul's formula for the Levi-Civita connection with respect to g, we obtain

$$\nabla_{e_1}e_3 = -\in e_1 + \frac{1}{2} \in e^{2z}e_2, \nabla_{e_2}e_3 = -\in e_2 - \frac{1}{2}\in e^{2z}e_1, \nabla_{e_3}e_3 = 0,$$

$$\nabla_{e_1}e_2 = -\frac{1}{2}\in e^{2z}e_3, \nabla_{e_2}e_2 = \in e_3 + \in ye^z e_1, \nabla_{e_3}e_2 = -\frac{1}{2}\in e^{2z}e_1,$$

$$\nabla_{e_1}e_1 = \in e_3, \nabla_{e_2}e_1 = -\in ye^z e_2 + \frac{1}{2}\in e^{2z}e_3, \nabla_{e_3}e_1 = \frac{1}{2}\in e^{2z}e_2.$$

Now, for $e_3 = \xi$, above results satisfy

$$\nabla_X \xi = \in \{-\alpha\phi X + \beta(X - \eta(X)\xi)\},$$

with $\alpha = -\frac{1}{2}e^{2z}$ and $\beta = -1$. Consequently $M(\phi, \xi, \eta, g, \in)$ is a 3-dimensional $\in$ -Trans-Sasakian manifold[10].

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_1, e_2)e_2 = -(1 + \frac{3}{4}e^{4z} + y^2 e^{2z})e_1, R(e_1, e_2)e_1 = (1 + \frac{3}{4}e^{4z} + y^2 e^{2z})e_2$$

$$R(e_2, e_3)e_3 = -e^{2z}e_1 + (\frac{1}{4}e^{4z} - 1)e_2, R(e_1, e_3)e_3 = e^{2z}e_2 + (\frac{1}{4}e^{4z} - 1)e_1,$$

$$R(e_1, e_3)e_2 = -e^{2z}e_3, R(e_1, e_3)e_1 = -(1 + \frac{1}{4}e^{4z})e_3,$$

$$R(e_2, e_3)e_2 = ye^z e_1 + (1 - \frac{1}{4}e^{4z})e_3, R(e_3, e_1)e_1 = (1 + \frac{1}{4}e^{4z})e_3,$$

$$R(e_3, e_2)e_1 = -e^{2z}e_3 + ye^z e_2.$$

and the components which can be obtained from these by the symmetry properties.

Using the components of the curvature tensor, we can easily calculate the non –vanishing components of the Ricci tensor S and its covariant derivatives as follows:

$$S(e_1,e_1) = \in (-\frac{1}{2}e^{4z} - y^2 e^{2z}), S(e_2,e_2) = \in (-2-\frac{1}{2}e^{4z} - y^2 e^{2z}), S(e_3,e_3) = \in (\frac{1}{2}e^{4z} - 2)$$

$$\nabla_{e_1} S(e_1,e_3) = (-\frac{1}{2}e^{4z} - y^2 e^{2z}) + \in (2 - \frac{1}{2}e^{4z}), \nabla_{e_1} S(e_1,e_1) = \nabla_{e_1} S(e_2,e_2) = 0,$$

$$\nabla_{e_1} S(e_2,e_3) = \frac{1}{2}(e^{6z} + y^2 e^{4z}), \nabla_{e_1} S(e_1,e_2) = \nabla_{e_2} S(e_2,e_2) = 0,$$

$$\nabla_{e_2} S(e_1,e_3) = e^{2z} - \frac{1}{2}(e^{6z} - y^2 e^{4z}), \nabla_{e_2} S(e_3,e_3) = \nabla_{e_3} S(e_1,e_1) = 0,$$

$$\nabla_{e_2} S(e_2,e_3) = -(e^{4z} + y^2 e^{2z}), \nabla_{e_2} S(e_1,e_1) = \nabla_{e_3} S(e_2,e_2) = 0,$$

$$\nabla_{e_3} S(e_1,e_2) = e^{2z}, \nabla_{e_3} S(e_1,e_3) = \nabla_{e_3} S(e_2,e_3) = 0.$$

Since $\{e_1, e_2, e_3\}$ is an orthonormal basis of (M^3, g), any vector X and Y can be written as

$$X = a_1 e_1 + a_2 e_2 + a_3 e_3, Y = b_1 e_1 + b_2 e_2 + b_3 e_3,$$

Where $a_i, b_i (i = 1, 2, 3)$ are positive real numbers. Now

$$S(X,Y) = a_1 b_1 S(e_1,e_1) + a_2 b_2 S(e_2,e_2) + a_3 b_3 S(e_3,e_3)$$
$$+ (a_1 b_2 + a_2 b_1) S(e_1,e_2) + (a_1 b_3 + a_3 b_1) S(e_1,e_3)$$
$$+ (a_2 b_3 + a_3 b_2) S(e_2,e_3)$$
$$= - \in a_1 b_1 (\frac{1}{2}e^{4z} + y^2 e^{2z}) - \in a_2 b_2 (2 + \frac{1}{2}e^{4z} + y^2 e^{2z})$$
$$+ \in a_3 b_3 (\frac{1}{2}e^{4z} - 2)$$

from (1.8) we get

$$P(X,Y) = S(X,Y) - \frac{r}{3} g(X,Y)$$

$$= - \in a_1 b_1 (\frac{1}{2}e^{4z} + y^2 e^{2z}) - \in a_2 b_2 (2 + \frac{1}{2}e^{4z} + y^2 e^{2z})$$

$$+ \in a_3 b_3 (\frac{1}{2} e^{4z} - 2) - \frac{r}{3} (a_1 b_1 + a_2 b_2 + a_3 b_3) \in = \rho_1, \text{ say.}$$

We choose $a_i, b_i (i = 1, 2, 3)$ in such way that $P(X, Y) = \rho_1 \neq 0$. The covariant derivatives of the Ricci tensor $S(X, Y)$ are given by

$$\nabla_{e_1} S(X, Y) = a_1 b_1 \nabla_{e_1} S(e_1, e_1) + a_2 b_2 \nabla_{e_1} S(e_2, e_2) + a_3 b_3 \nabla_{e_1} S(e_3, e_3)$$

$$+ (a_1 b_2 + a_2 b_1) \nabla_{e_1} S(e_1, e_2) + (a_1 b_3 + a_3 b_1) \nabla_{e_1} S(e_1, e_3)$$

$$+ (a_2 b_3 + a_3 b_2) \nabla_{e_1} S(e_2, e_3)$$

$$= (a_1 b_3 + a_3 a_1) \{ \in (2 - \frac{1}{2} e^{4z}) - (\frac{1}{2} e^{4z} + y^2 e^{2z}) \}$$

$$+ \frac{1}{2} (a_2 b_3 + a_3 b_2)(e^{6z} + y^2 e^{4z})$$

$$\nabla_{e_1} P(X, Y) = \nabla_{e_1} S(X, Y) - \nabla_{e_1} [\frac{r}{3} g(X, Y)]$$

$$= (a_1 b_3 + a_3 a_1) \{ \in (2 - \frac{1}{2} e^{4z}) - (\frac{1}{2} e^{4z} + y^2 e^{2z}) \}$$

$$+ \frac{1}{2} (a_2 b_3 + a_3 b_2)(e^{6z} + y^2 e^{4z}) - \frac{[\nabla_{e_1}(r)]}{3} (a_1 b_1 + a_2 b_2 + a_3 b_3) \in$$

$$= \rho_2, \text{ say.}$$

$$\nabla_{e_2} P(X, Y) = -(a_1 b_2 + a_2 b_1)(2y e^z) + (a_1 b_3 + a_3 b_1)[e^{2z} - \frac{1}{2}(e^{6z} + y^2 e^{4z})]$$

$$-(a_2 b_3 + a_3 b_2)(e^{4z} + y^2 e^{2z}) - \frac{[\nabla_{e_2}(r)]}{3}(a_1 b_1 + a_2 b_2 + a_3 b_3) \in = \rho_3, \text{ say.}$$

$$\nabla_{e_3} P(X, Y) = (a_1 b_2 + a_2 b_1)(e^{2z}) - \frac{[\nabla_{e_3}(r)]}{3}(a_1 b_1 + a_2 b_2 + a_3 b_3) \in = \rho_4, \text{ say.}$$

Using (1.9), we get

$$\nabla_{e_1} P(X, Y) = A(e_1) P(X, Y) + B(X) P(e_1, Y) + D(Y) P(X, e_1) \qquad (6.13)$$

$$\nabla_{e_2} P(X, Y) = A(e_2) P(X, Y) + B(X) P(e_2, Y) + D(Y) P(X, e_2) \qquad (6.14)$$

$$\nabla_{e_3} P(X, Y) = A(e_3) P(X, Y) + B(X) P(e_3, Y) + D(Y) P(X, e_3) \qquad (6.15)$$

Setting,

$$A(e_1) = \frac{\rho_2}{\rho_1}, A(e_2) = \frac{\rho_3}{\rho_1}, A(e_3) = \frac{\rho_4}{\rho_1},$$

(6.16)

from (6.13),(6.14) and (6.15) it is easy to see that,

$$B(X)P(e_1, Y) + D(Y)P(X, e_1) = 0 \quad (6.17)$$

$$B(X)P(e_2, Y) + D(Y)P(X, e_2) = 0 \quad (6.18)$$

$$B(X)P(e_3, Y) + D(Y)P(X, e_3) = 0 \quad (6.19)$$

From these homogeneous equations in B and C, one obtains non trivial solutions

$$B(e_1) = -D(e_1)$$

Hence we can take arbitrarily as,

$$B(e_1) = (a_2 b_3 - a_3 b_2)e^{4z}, D(e_1)$$

$$= (a_3 b_2 - a_2 b_3)e^{4z} \quad (6.20)$$

Also from (6.17),(6.18) and (6.19), we find

$$B(e_2) = 0, D(e_2) = 0, B(e_3) = 0, D(e_3) = 0,$$

(6.21)

The 1-form given by (6.16),(6.20) and (6.21), the manifold under consideration is a weakly Concircular Ricci symmetric ∈- Trans-Sasakian manifold. This leads to the following:

Theorem 6.8: There exists a ∈- Trans-Sasakian manifold which is weakly concircular Ricci symmetric but neither Ricci symmetric nor Ricci-recurrent.

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