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An Introduction of Exact Sequences and Exact Functor

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Abstract

Short exact sequence are fundamental objects in abelian categories, and one of the most basic ways to study an additive functor from one abelian category to another is to examine what it does to short exact sequences.

Introduction

In the decade of the 40's, the invasion was reversed and topology invaded algebra. Among the principal names here were Eilenberg, MacLane, Hochschild Kosul. This created "homological algebra" and first deeply influential book was in fact called "Homological Algebra" and another by H. Cartan and S. Eilengerg (1956). Homological Algebra has now reached into almost every corner of modern Mathematics.

Preliminaries:

We must first introduce some basic concept. Throughout, R will denote a commutative ring. For the sake of discussion, one may assume either that R is a field or that R = z.

Chain Complexes:

Let R = be commutative ring with 1. A chain complex is a collection of $\{C_i\}_{i\in z}$ of R-modules and maps $\{d_i : C_i \to C_{i-1}\}$ called differentials such that $d_i.d_{i-1} = 0$, [similarly, a chain complex is collection of $\{C^i\}_{i\in z}$ of R-modules and maps] $\{d^i : C^i \to C^{i+1}\}$ such that $d^{i+1}.\ d^{i} = 0$.

$$. . . \to C_{i+1} \to C_i \to C_{i-1} \to . . .$$

Remark:

The only difference between a chain complex and co-chain complex is whether the map go up in degree (are of degree 1) or go down in degree (are of degree-1). Every chain complex is canonically a co-chain complex by setting $C^i = C_{-i}$ and $d^i = d_{-i}$.

We will be convenient to treat a special case first that of a differential module[2,3]. By a differential module A we understand an R-module A with an endomorphism d of square zero : $d^2 = 0$. The map in the category are taken to be homomorphism preserving the structure, *i.e.* the homomorphism $f : A \to B$ such that the square shown commutes:

$$\begin{array}{ccc} A & \xrightarrow{d} & A \\ \downarrow f & & \downarrow f \\ B & \xrightarrow{d} & B \end{array}$$

These maps traditionally called chain-maps (The condition $d^2=0$ means that $Imd \subseteq kerd$).

Definition:

Let $C = (C_i.d_i)$ and $C' = (C_i^{'} \,.\, d_i^{'})$ be two complexes. The chain map between them is a collection of maps $f = \{f_i : C_i \rightarrow C_i^{'}\}$ such that $d'_i.f_i = f_{i-1}.d_i$ *i.e.* the following diagram

$$\begin{array}{ccccccc} \cdots C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \xrightarrow{d_i} & C_{i-1} & \rightarrow \cdots \\ \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} & \\ \cdots C_{i+1}^{'} & \xrightarrow{d_{i+1}^{'}} & C_i^{'} & \xrightarrow{d_i^{'}} & C_{i-1}^{'} & \rightarrow \cdots \end{array}$$

is commutative for all $i \in z$.

Let C be chain complex. Let $Z_i = kerd_i$ be the cycles of C_i and $B_i = Imd_{i+1}$ be the boundaries of C_i. Since $d^2 = 0$ ($Imd \subseteq kerd$). We have that for each $i \in z$, $B_i \subseteq Z_i$. Finally $H_i\,(C) = Z_i / B_i$ $i \in z$ is called the *homotopy of* (Similarly, for a co-chain complex, we define the i^th *co-homology* $H^i(C)$).

Given a map $f : C \rightarrow C'$ between two chain complexes, f maps cycles to cycles and boundaries to boundaries, and thus, f induce a map $f_* = H(C) \rightarrow H(C')$.

It often happens[4,5] that two different chain maps induce the same maps on homology.

Definition:

Two chain maps $f.g : C \rightarrow C'$ are said to be chain *homotopic* written $f \sim g$, if there exist a collection $\{s_i\}_{i \in z}$ of chain maps $s_i : C_i \rightarrow C'_i$ *s.t.* $f - g = d'_{i+1}.s_i + s_{i-1}.d_i$ $\forall\ i \in \mathbf{z}$

Proposition[1] *:*

Let $f : C \rightarrow C'$ be a chain map of complexes, f maps cycles to cycles and boundaries to boundaries. Then f induce a map $f_* = H(C) \rightarrow H(C')$, $i \in z$.

Moreover (i) $id^* = id$ and (ii) If $g : C \rightarrow C'$ is another chain map of complexes, then $(g.f)_* = g_i^{*}.f_i^{*}$ $\forall\ i \in z$.

Proof:

Since f is a chain map, $f : C \rightarrow C'$ of complexes. We have to commute diagram

$$\begin{array}{ccccc} \cdots\ C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \cdots \\ \downarrow f_{i+1} & & \downarrow f_i & \\ \cdots\ C_{i+1}^{'} & \xrightarrow{d_{i+1}^{'}} & C_i^{'} & \cdots \end{array}$$

since $d'_{i+1}.f_{i+1} = f_i.d_{i+1}$ $\forall$ $i \in z$.

we have $f_{i+1}(ker\, d_{i+1}) \subseteq ker\, d'_{i+1}$ and $f_i(lmd_{i+1}) \subseteq lmd'_{i+1}$ $\forall$ i z.

Hence the chain map, $f_i^* = \dfrac{Z_i(C)}{B_i(C)} \rightarrow \dfrac{Z_i(C')}{B_i(C')}$ given by

$f_i^*(\alpha + B_i(c)) = f_i(\alpha) + B_i(C)$. The f_i^* is well defined and satisfies (i) and (ii).

Preposition : [1]

Let $f : C \to C'$ be a chain map of complexes such that ~*g*. Then $f_i^* = g_i^*$.

Proof : [1].

Definition:

Two complexes *C* and *C'* are said to be homotopically equivalence if there exist Chain maps $f : C \to C'$ and $g : C' \to C$ such that, $g = I_C$ and $g.f = I_{C'}$.

Corollary:

If the chain map $f : C \to C'$ is *homotopically* equivalence,

then $f_i^* = H_i(C) \to H_i(C')$ is an isomorphism for all $i \in \mathbb{Z}$.

Exact Sequences:

A sequence $A \to B \to C$ is said to be exact at B if and only if

ker($B \to C$) = *Im*($A \to B$).

A short exact sequence is an exact five term sequence which begins and ends with zero $0 \to A \to B \to C \to 0$.

A longer sequence for example : $\ldots \to A \to B \to C \to D \to E \to \ldots$ is said to be exact if and only if it is exact at module where this makes sense.

If *C* is a complex such that $H_i(C) = 0$, then *C* is said to be exact at position i. If is exact at all positions, then *C* is said to be exact sequence.

This means that the image of one map is equal to the kernel of next. We make the convention of saying that a sequence $A_n \to A_{n-1} \to \ldots \to A_1 \to A_0$ is exact if $H_i(A) = 0$
$0 < i < n$.

Example:

A complex of the form $0 \to A \to 0$ is exact if and only if $A \cong 0$. A complex of the form $0 \to A \to B \to 0$ is exact iff the middle map is isomorphism.

Proposition:

Let $f : A \to B$ be a map of complexes such that

(i) $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if *f* is injective.

(ii) $A \xrightarrow{f} B \longrightarrow 0$ is exact if and only if *f* is surjective.

A pair of homomorphism of homomorphisms $C' \xrightarrow{f} C \xrightarrow{g} C''$ is exact at *C* if *Imf* = ker *g*.

A sequence $\ldots \to C_{i+1} \to C_i \to C_{i-1} \to \ldots$ is exact if it is exact at every C_i that is between two homomorphisms.

We can consider[1-3] not only short exact sequence of modules, but also short exact sequence of chain complexes, namely a commutative diagrams of the from

$$\begin{array}{ccccccc}
\ldots \to & A_{i+1} & \to & A_i & \to & A_{i-1} & \to \ldots \\
& \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} & \\
\ldots \to & B_{i+1} & \to & B_i & \to & B_{i-1} & \to \ldots \\
& \downarrow g_{i+1} & & \downarrow g_i & & \downarrow g_{i-1} & \\
\ldots \to & C_{i+1} & \to & C_i & \to & C_{i-1} & \to \ldots
\end{array}$$

Such that rows are chain complexes and $0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ is an exact sequence.

Exact Functor:

Short exact sequence are fundamental objects in abelian categories, and one of the most basic ways to study an additive functor from one abelian category to another is to examine what it does to short exact sequences.

Let F be a covariant functor and let G be a contravariant functor.

We say that F (resp. G) is exact if,

$F : 0 \to F(A) \to F(B) \to F(C) \to 0$ and ($G : 0 \to G(A) \to G(B) \to G(C) \to 0$)

is exact sequence, whenever $0 \to A \to B \to C \to 0$.

Similarly, we say that F (resp. G) is left exact, if $F : 0 \to F(A) \to F(B) \to F(C) \to 0$ and (G: $0 \to G(A) \to G(B) \to G(C) \to 0$) is exact sequence whenever $0 \to A \to B \to C \to 0$.

Finally, we say the F (resp. G) is right exact, if $F : 0 \to F(A) \to F(B) \to F(C) \to 0$ and ($G : 0 \to G(A) \to G(B) \to G(C) \to 0$) is exact Sequence, whenever $0 \to A \ \ B \to C \to 0$.

Example :

Let M be an R-module. Then the functor,

$hom_R(M, -) : R - mod \to R\text{–}mod'N \to hom_R(M, N)$ and

$hom_R(- , M) : R - hom^{op} \to R - mod'N \to hom_R(N, M)$

are both left exact, the first being covariant and the second being contravariant.

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