Ultra Scientist Vol. 24(1)A, 153-156 (2012).

Weakly compatible maps of a complete metric space

ANIL RAJPUT[*], PRADEEP KUMAR DWIVEDI[**], RAMESH PRASAD AHARWAL [***] and ARCHNA JAIN[****]

[**]CSA, Govt. P.G. College, Sehore (INDIA)
(email: dranilrajput@hotmail.com)
[***]Millennium Group of Institution, Bhopal (INDIA)
(pkdwivedi76@gmail.com) Mobial 9993960790
[***]Govt. P.G. College Bareli (M.P.) (INDIA)
(ramesh_ahirwal_neetu@yahoo.com)
[****]Govt. P. G. College BHEL, Bhopal (INDIA)

(Acceptance Date 7th February, 2012)

Abstract

The purpose of this paper is to prove a common fixed point theorem, from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity, which generalizes the results of Jungck[3], Fisher[1], Kang and Kim[8], Jachymski[2], and Rhoades[9].

Introduction

In 1976, Jungck[4] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that, let (X, d) be a complete metric space. If T satisfies

$$d(Tx, Ty) \le kd(x, y)$$

for each $x, y \in X$ where $0 \le k < 1$, then T has a unique fixed point in X. On the other hand Sessa[10], defined weak commutatively and proved common fixed point theorem for weakly commuting maps. Further Jungck[5], introduced more generalized commutatively, the so-called compatibility, which is more general than that of weak commutatively[3].

It has been known from the paper of Kannan[7] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. In 1998, Jungck and Rhoades[6], introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but converse need not be true, In this paper, we prove a fixed point theorem for weakly compatible maps without appeal to continuity, which generalizes the result of Fisher[1], Jachymski[2], Kang and Kim[8] and Rhoades *et al*.[9].

Fixed Point Theorem :

Let R+ denote the set of non-negative heal numbers and F is a family of all mappings $\psi\,(R^+)^6 \to R^+$ such that is ψ upper semi continuous, non decreasing in each coordinate variable and for any t > 0,

$$\psi(t) = \varphi\left(t, t, 0, 2t, \frac{1}{2}t, t\right) < t$$

Let P, Q, S and T be mappings from a metric space (X,d) into is self satisfying the following conditions.

$$P(X) \subset T(X) \text{ and } Q(X)\ S(X) \quad (1)$$

$$d(Px, Qy) \le \varphi\ \{(Sx, Ty), d(Px, Sx), d(Qy, Ty), d(Px,Ty), d(Sx,Qy), \frac{1}{2}[d(Px,Ty)+d(Sx, Qy)]$$

for all x, $y \in X$, where $\varphi \in F$ (2)

Theorem: Let (P, S) and (Q, T) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (1) and (2). Then P, Q, S and T have a unique common fixed point in X.

Proof. Let $\{y_n\}$ is a Cauchy sequence in X, Since X is complete there exists a point z in X such that

$$\lim_{n\to\infty} y_n = z,\ \lim_{n\to\infty} P_{2n} = \lim_{n\to\infty} T_{2n+1} = z \text{ and}$$

$$\lim_{n\to\infty} Q_{2n+1} = \lim_{n\to\infty} S_{2n+2} = z$$

i.e.,

$$\lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Since Q (X) $\subset$ S(X) there exists a point $u \in X$.

Such that z= Su. Then using (2),

$$d(Pu, z) = d(Pu, Qx_{2n+1})$$

$$\le \varphi\{d(Su, Tx_{2n+1}), d(Pu, Su), d(Qx_{2n+1}, T_{2n+1}), d(Pu, Tx_{2n+1})$$

$$d(Su, Qx_{2n+1}), \frac{1}{2}[d(Pu, Tx_{2n+1}) + d(Su, Qx_{2n+1})]\}$$

Taking limit as $n \to \infty$, we get

$$d(Pu, z) \le \varphi\ \{d(z, z), d(Pu, z), d(z, z), d(Pu, z)\ d(z, z), \frac{1}{2}[d(Pu, z) + d(z, z)]\}$$

$$= \varphi\ \{0, d(Pu, z), 0, d(Pu, z), 0, \frac{1}{2}d(pu, z)\} < d(Pu, z)$$

Therefore z = Pu = Su

Since $P(X) \subset T(X)$ there exists a point $v \in X$ such that z = Tv. Then again using (2),

$$d(z, Qv) = d(Pu, Qv).$$

$$\le \varphi\{d(Su, Tv), d(Pu, Su), d(Qv, Tv), d(Pu, Tv)\ d(Su, Qv), \frac{1}{2}[d(Pu, Tv) + d(Su, Qv)]\}$$

$$= \varphi\ \{d(z, z), d(z, z), d(Qv, z), d(z, z), d(z, Qv), \frac{1}{2}[d(z, z) + d(z, Qv)]\}$$

$$= \varphi\{0, 0, d(Qv, z), 0, d(z, Qv), \frac{1}{2}d(z,Qv)\} < d(z, Qv)$$

Therefore z = Qv = Tv Thus Pu = Su = Qv =

Tv= z. Since pair of maps P and S are weakly compatible, then PSu = SPu i.e. Pz = Sz. Now we show that z is a fixed point of P. If Pz $\neq$ z, then by (2),

$$d(Pz, z) = d(Pz, Qv)$$
$$\leq \varphi\{d(Sz, Tv), d(Pz, Sz), d(Qv, Tv), d(Pz, Tv), d(Sz, Qv), \frac{1}{2}[d(Pz, Tv) + d(Sz, Qv)]\}$$
$$= \varphi\{d(Pz, z), d(Pz, Pz), d(z, z), d(Pz, z)\ d(Pz, z), \frac{1}{2}[d(Pz, z) + d(pz, z)]\}$$
$$= \varphi\{d(Pz, z), 0, 0, d(Pz, z), d(Pz, z), d(Pz, z)\}$$
$$< d(Pz, z)$$

Therefore Pz = z Hence Pz = Sz = z

Similarly, Pair of maps Q and T are weakly compatible, we have Qz = Tz= z, then by (2)

$$d(z, Qz) = d(Pz, Qz)$$
$$\leq \varphi\{d(Sz, Tz), d(Pz, Sz), d(Qz, Tz), d(Pz, Tz), d(Sz, Qz), \frac{1}{2}[d(Pz, Tz) + d(Sz, Qz)]\}$$
$$= \varphi\{d(z, Qz), d(z, z), d(Qz, Qz), d(z, Qz), d(z, Qz), \frac{1}{2}[d(z, Qz) + d(z, Qz)]\}$$
$$= \varphi\{d(z, Qz), 0, 0, d(z, Qz), d(z, Qz), d(z, Qz)\}$$
$$< d(z, Qz)$$

There fore Qz = z. Hence Qz = Tz = z.

Thus z= Pz = Qz = Sz= Tz and z is a common fixed paint of P, Q, S and T.

Finally in order to prove the uniqess of z suppose that z and w, z $\neq$ w are common fixed point of P, Q, S and T. Then by (2) we obtain

$$d(z, w) = d(Pz, Qw)$$
$$\leq \varphi\{d(Sz, Tw), d(Pz, Sz), d(Qw, Tw), d(Pz, Tw), d(Sz, Qw), \frac{1}{2}[d(Pz, Tw) + d(Sz, Qw)]\}$$
$$d(z, w) \leq \varphi\{d(z, w), d(z, z), d(w, w), d(z, w), d(z, w), \frac{1}{2}[d(z, w) + d(z, w)]\}$$
$$= \varphi\{d(z, w), 0, 0, d(z, w), d(z, w), d(z, w), d(z, w)\}$$
$$< d(z, w)$$

Therefore z = w.

Hence z is a unique common fixed point of the mappings P, Q, S and T.

The following corollaries follow immediately from our theorem.

Corollary 1 :

Let (P, S) and Q, T) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (3.1), (3.3) and (3.10)

$d(Px, Qy) \leq h\, M(x, y)$, $0 \leq h < 1$, $x, y, \in X$, where

$$M(x, y) = \max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty), \frac{1}{2} d(P, Ty), \frac{1}{2} d(S, Qy), \quad d(P, Ty)\} \qquad (3)$$

Then P, Q, S, and T have a unique common fixed point in X.

Proof : We consider the function $\varphi : [0, \infty]^6 \to [0, \infty]$

defined by

φ ($x_1, x_2, x_3, x_4, x_5, x_6$) = h max $\{x_1, x_2, x_3, \frac{1}{2} x_4, \frac{1}{2}, x_5, x_6\}$

Since $\varphi \in F$, we can apply in our theorem and deduce the corollary.

Corollary 2 :

Let (P, S) and (Q, T) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (1), (3) and (3)

$$d(Px, Q_y) \leq h \max \{d(Px, Sx), d(Q_y, T_y), \frac{1}{2} d(Px, T_y) \frac{1}{2} d(Sx, Q_y), \frac{1}{2} [d(Px, T_y) + d(Sx, Q_y)], d(Sx, T_y)\} \quad (3.11)$$

for all x, y in X, where $0 \leq h < 1$

Proof : We take same process as above Theorem.

References

1. Fisher B., Common fixed points of four mappings, *Bull. Inst. Math. Acad. Sci. 11*, 103-113 (1983).
2. Jachymski J., Common fixed point theorems for some families of maps, *J. Pure Appl. Math. 25*, 925–937 (1994).
3. Jungck G., Compatible mappings and common fixed points (2), *Int. J. Math. Math. Sci. 11*, 285-288 (1988).
4. Jungck G., Commuting maps and fixed points, *Am. Math. Mon. 83*, 261 (1976).
5. Jungck G., Compatible mappings and common fixed points, *Int. J. Math. Math. Sci. 9*, 771-779 (1986).
6. Jungck G. and Rhoades B. E., Fixed point for set valued functions without continuity, *Indian. J. Pure Appl. Math. 29(3),* 227-238 (1998).
7. Kannan R., Some results on fixed points, *Bull. Cal. Math. Soc. 60,* 71-76 (1968).
8. Kang S. M. and Kim Y. P., Common fixed points theorems. *Math. Japonica 37(6),* 1031-1039 (1992).
9. Rhoades B.E., Park S. and Moon K. B., On generalizations of the Meir-Keeler type contraction maps';' *Math. Anal. Appl. 146*, 482 (1990).
10. Sessa S., On a weak commutatively condition of mappings in fixed point considerations, *Pub. Inst. Math. 32(46)*, 149-153 (1982).