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Cost benefit analysis of 2-out of 3: G system with correlated failure and repair times

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Abstract

The paper analyses a stochastic model of a cold standby system consisting of three non identical units-A, B and C in which two units are operative at a time. A single repair facility is considered to repair a failed unit. The priority in operation is being given to unit A over to unit B and to unit B over unit C. In case of repair of failed units the priority is just reverse than operation. The failure and repair times of each unit are assumed to follow bivariate exponential distribution. Several measures of system effectiveness are obtained by using regenerative point technique.

Key words: Redundant, Stochastic, priority unit, Cold standby, Regenerative point, Bivariate exponential, Reliability, MTSF, Expected uptime, Busy period, Cost benefit analysis.

1. Introduction

A large number of authors have analysed two-unit redundant system models including Gupta and Shivakar[7]. Gupta *et al*.[6] have studied the cost benefit analysis of a single server three-unit redundant system model with inspection, delayed replacement and two types of repair. Failure time distributions are taken negative exponential whereas all the other time distributions are assumed to be general. Gupta *et al*.[8] investigated a three-unit system model having super-priority (sp), priority (p) and ordinary (o) units. The preference in operations and repair is being provided to sp unit over p and o units and to p unit over the o unit. To obtain various reliability measures, failure and repair time distributions of each unit are taken general. In the analysis of all the above system models, the common assumption is that the failure and repair times are considered to be uncorrelated random variables. The concept of correlated failure and repair times has already been introduced by Goel *et al*.[3,4] to analysis two unit system models.

The purpose of the present paper is to investigate a three-unit system model with correlated failure and repair times. The system description and assumptions are as follows-

(i) The system consists of three non-identical units- A, B and C. Initially, the unit A and B are operative and units C is kept as cold standby.

(ii) The switching device used to put the leading standby unit into operation is always perfect and instantaneous.

(iii) The preference in operation is being given to unit A over the units B and C and to unit B over the unit C.

(iv) A single repairman is considered to repair a failed unit. The preference in repair is being given to unit C over B and A and to unit B over the unit A *i.e.* the unit's order of preference in repair is quite reverse than the order considered in operation.

(v) The joint distribution of failure and repair times for each unit is taken to be bivariate exponential having the density function:

$$f_i(x,y)=\alpha_i\,\beta_i\,(1-r_i)\,e^{-(\alpha_i x+\beta_i y)}$$

$$I_0\left(2\sqrt{\alpha_i\beta_i r_i xy}\right)\;;\; i=1,2,3$$

$$x,\; y,\; \alpha_i,\beta_i > 0\;;\; 0\le r_i\le 1,$$

where $I_0(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{(k!)^2}$

is the modified Bessel function of type I and order zero.

(vi) If unit B fails during the repair of unit A then unit B is taken up for repair discontinuing the repair of unit A. The residual repair time of unit A is supposed to be independently distributed having the negative exponential distribution with parameter β_1. Similar action is taken if unit C fails during the repair of unit B. The parameter of the residual repair time distribution of unit B is β_2.

Using the techniques of regenerative point, the following characteristics of interest to system designers and operations managers are obtained:

(i) The reliability of the system and mean time to system failure (MTSF).

(ii) Expected working (up) time of units A, B and C during time interval (0, t] and in steady state).

(iii) Expected busy period of repairman in repairing of units A, B and C during (0, t] and in steady state. Cost Benefit Analysis of a Three-Unit System with Correlated Failure and Repair Times.

(iv) Net expected profit earned by the system during interval (0, t) and in steady state.

2. Nomenclature :

X_i, Y_i Random variables denoting the failure and repair times for units A, B and C respectively for i = 1, 2, 3.

f_i- (x, y) Joint p.d.f. of X_i, Y_i

g_i (x) Marginal p.d.f. of X_1 = x

$$=\alpha_i(1-r_i)\exp\left[-\alpha_i(1-r_i)x\right],\; i=1,2,3$$

k_i (y/x) Conditional p.d.f. of Y_i given X_i = x

$$=\beta_i\exp\left[-(\beta_i y+\alpha_i r_i x)\right]I_0\left(2\sqrt{\alpha_i\,\beta_i\,r_i\,xy}\right)$$

q_{ij}-(.) pdf. of direct transition from regenerative state S_i to S_j.

p_{ij} steady state direct transition probability from state S_i to S_j such that

$$p_{ij}=\int_0^{\infty}q_{ij}(u)\,du$$

$q_{ij}^{(u,v,w)}(.)$ pdf. of transition time from regenerative state S_i to S_j via non-regenerative states S_u, S_v and S_w.

$p_{ij}^{(u,v,w)}$ Steady state transition probability from regenerative state S_i to S_j via non-regenerative states S_u, S_v and S_w such that

$$p_{ij}^{(u,v,w)} = \int_0^\infty q_{ij}^{(u,v,w)}(z)dz = p_{iu}\, p_{uv}\, p_{vw}\, p_{wj}$$

$p_{ij/x}$ Steady state probability of transition from non-regenerative state S_i to regenerative state S_j given that the unit under repair in state S_i entered into F-mode after an operation of time x.

$Z_i(t)$ Probability that the system sojourns in state S_i upto time t.

μ_i Mean sojourn time in state $S_i = \int_0^\infty Z_i(t)\,dt = \lim_{s\to 0} Z_i(s)$

$\mu_{i/x}$ Mean sojourn time in non-regenerative state S_i given that the unit under repair in this state entered into F-mode after an operation of time x.

© Symbols for convolution

$$A(t) \circledcirc B(t) = \int_0^t A(u)\, B(t-u)\, du$$

Symbols for the states of the system:

A_o, B_o, C_o unit A, B, C is operative.

A_s, B_s, C_s unit A, B, C is in standby.

A_r, B_r, C_r unit A, B, C is under repair.

A_w, B_w, C_w unit A, B, C is waiting for repair.

A_r, B_r, repair of unit A, B is re-started.

A_w, B_w unit A, B is waiting to be re-started repair.

Using these symbols, the states of the system and the transitions between them along with the transition time/repair rates are shown in Fig. 1. The epochs of the transitions from S_0 to S_1, S_3, S_1 to S_2, S_4 and S_3 to S_2, S_5 are non-regenerative.

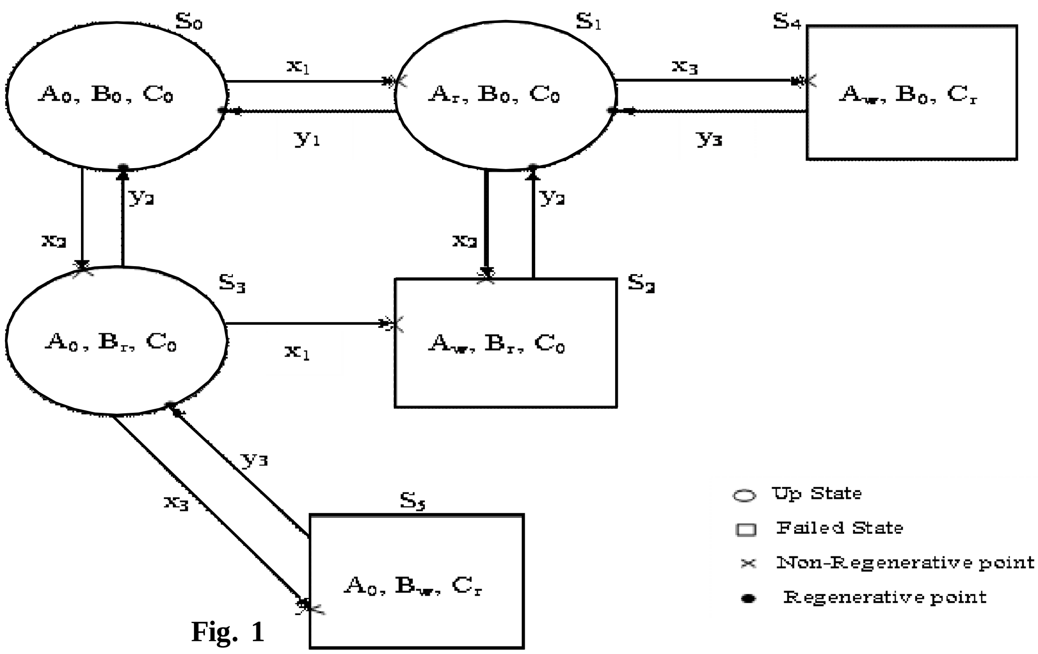

Fig. 1

3. Transition probabilities and Sojourn Times:

By definition and simple probabilistic arguments the direct or one step unconditional transition probabilities can be obtained as follow; Similarly the conditional probabilities can be obtained as follows.

$$p_{01} = \int_0^\infty \alpha_1(1-r_1)e^{-\alpha_1(1-r_1)u}e^{-\alpha_2(1-r_2)}du = \frac{\alpha_1(1-r_1)}{\alpha_1(1-r_1)+\alpha_2(1-r_2)}$$

$$p_{03} = \int_0^\infty \alpha_2(1-r_2)e^{-\alpha_2(1-r_2)u}e^{-\alpha_1(1-r_1)u}du = \frac{\alpha_2(1-r_2)}{\alpha_1(1-r_1)+\alpha_2(1-r_2)}$$

So that, $p_{01}+p_{03} = 1$ (3.1)

$$p_{10} = \int_0^\infty \beta_1 e^{-\beta_1 u}e^{-\alpha_2(1-r_2)u}e^{-\alpha_3(1-r_3)u}du = \frac{\beta_1}{\beta_1+\alpha_2(1-r_2)+\alpha_3(1-r_3)}$$

$$p_{12} = \int_0^\infty \alpha_2(1-r_2)e^{-\alpha_2(1-r_2)u}\ e^{-\beta_1 u}\ e^{-\alpha_3(1-r_3)u}.du = \frac{\alpha_2(1-r_2)}{\beta_1+\alpha_2(1-r_2)+\alpha_3(1-r_3)}$$

$$p_{14} = \int_0^\infty \alpha_3(1-r_3)e^{-\alpha_3(1-r_3)u}\ e^{-\beta_1 u}\ e^{-\alpha_2(1-r_2)u}du = \frac{\alpha_3(1-r_3)}{\beta_1+\alpha_2(1-r_2)+\alpha_3(1-r_3)}$$

So that, $p_{10}+p_{12}+p_{14}=1$ (3.2)

$$p_{30} = \int_0^\infty \beta_2 e^{-\beta_2 u}\ e^{-\alpha_1(1-r_1)u}\ e^{-\alpha_3(1-r_3)u}du = \frac{\beta_2}{\beta_2+\alpha_1(1-r_1)+\alpha_2(1-r_2)}$$

$$p_{32} = \int_0^\infty \alpha_1(1-r_1)e^{-\alpha_1(1-r_1)u}\ e^{-\beta u}\ e^{-\alpha_3(1-r_3)u}.du = \frac{\alpha_2(1-r_1)}{\beta_2+\alpha_1(1-r_1)+\alpha_3(1-r_3)}$$

$$p_{35} = \int_0^\infty \alpha_3(1-r_3)e^{-\alpha_3(1-r_3)u}\ e^{-\beta u}\ e^{-\alpha_1(1-r_1)u}du = \frac{\alpha_3(1-r_3)}{\beta_2+\alpha_1(1-r_1)+\alpha_3(1-r_3)}$$

So that, $p_{30}+p_{32}+p_{35}=1$ (3.3)

$$p_{41|x} = \int_0^\infty \beta_3 e^{-(\beta_3 u+\alpha_3 r_{3x})}\sum_{j=p}^\infty \frac{(\alpha_3\beta_3 r_3 xu)^j}{(J!)^2}du = 1$$

$$p_{21|x} = \int_0^\infty \beta_2 e^{-(\beta_2 u+\alpha_2 r_2 x)}\sum_{j=p}^\infty \frac{(\alpha_2\beta_2 r_2 xu)^j}{(J!)^2} = 1$$

$$p_{53|x} = \int_0^\infty \beta_3 e^{-(\beta_3 u+\alpha_3 r_3 x)} \sum_{j=0}^\infty \frac{(\alpha_3\beta_3 r_3 xu)^j}{(J!)^2} du = 1$$

So that, $p_{14}=p_{21}=p_{53}=1$ (3.4)

Now the transition probability via one or more non-regenerative states one given by

$p_{00}^{(1)} = p_{10}p_{01}$, $\quad p_{00}^{(3)} = p_{03}p_{30}$

$p_{11}^{(2)} = p_{12}p_{21} = p_{12}$, $\quad p_{33}^{(5)} = p_{35}p_{53} = p_{35}$

$p_{11}^{(4)} = p_{14}p_{41} = p_{14}$, $\quad p_{31}^{(2)} = p_{32}p_{21} = p_{32}$

It can be easily varified that,

$$p_{33}^{(5)} + p_{31}^{(2)} = 1 - p_{30}, \qquad p_{11}^{(2)} + p_{11}^{(4)} = 1 - p_{10} \tag{3.5-3.6}$$

The mean Sojour times in various states are as follows :-

$$\mu_0 = \int_0^\infty e^{-\alpha_1(1-r_1)u} e^{-\alpha_2(1-r_2)u} du = \left[\frac{1}{\alpha_1(1-r_1)+\alpha_2(1-r_2)}\right] \tag{3.7}$$

$$\mu_1 = \int_0^\infty e^{-\beta_1 u}.e^{-\alpha_3(1-r_3)u} e^{-\alpha_2(1-r_2)u} du = \frac{1}{\beta_1+\alpha_2(1-r_2)+\alpha_3(1-r_3)} \tag{3.8}$$

$$\mu_3 = \int_0^\infty e^{-\beta_2 u}.e^{-\alpha_3(1-r_3)u} e^{-\alpha_1(1-r_1)u} du = \frac{1}{\beta_2+\alpha_1(1-r_1)+\alpha_3(1-r_3)} \tag{3.9}$$

$$\mu_{4|x} = \int_0^\infty e^{-\alpha_2(1-r_2)} \int_0^u \beta_3 e^{-(\beta_3 y+\alpha_3 r_3 x)} \sum_{j=0}^\infty \frac{(\alpha_3\beta_3 r_3 xy)}{(j!)^2},$$

$$\mu_4 = \int_0^\infty \mu_{4/x} g_3(x)dx = \frac{1}{\alpha_3(1-r_3)} \frac{(1-\beta_3^{'})}{(1-r_3\beta_3^{'})} \quad \text{where} \quad \beta'_3 = \frac{\beta_3}{\beta_3+\alpha_1(1-r_1)} \tag{3.10}$$

$$\mu_{2|x} = \int_0^\infty e^{-\alpha_3(1-r_3)u} \int_u^\infty \beta_2 e^{-(\beta_2 y+\alpha_2 r_2 x)} \sum \frac{(\alpha_3\beta_3 r_3 xy)^j}{(j!)^2}.dy.du$$

$$= \frac{1-\beta_2^{'} e^{-\alpha_2 r_2 x\left(1-\beta_2^{'}\right)}}{\alpha_3(1-r_3)}, \quad \text{where} \quad \beta_2 = \frac{\beta_2}{\beta_2+\alpha_3(1-r_3)}$$

$$\mu_2 = \frac{1-\beta_2'}{\alpha_3(1-r_3)(1-r_2\beta_2')} \quad (3.11)$$

Similarly, we can find

$$\mu_5 = \frac{1-\beta_3}{\alpha_1(1-r_1)(1-r_3\beta_3')}, \quad (3.12)$$

4. Analysis of Reliability out MTSF :

Let T_i be the time to system failure when at time t=0 it starts from regenerative state S_i. Then reliability of system is given by

$$R_i(t)=P(T_i > t)$$

To determine the reliability of the system, we assume the failed states S_2, S_4 and S_5 are absorbing states. By using simple probabilistic arguments, one can easily develop the recurrence among $R_i(t)$; i=0,1,3.Taking the Laplace Transform of the relation and simplifying the resulting set of algebraic equation for $R_0^*(s)$, we get after omitting the arguments 's' for brevity.

$$R_0^*(s) = \frac{Z_0^* + q_{01}Z_1^* + q_{03}^*Z_3^*}{\left(1-q_{00}^{(1)*} - q_{00}^{(3)*} - q_{01}^*q_{10}^* - q_{03}^*q_{30}^*\right)} \quad (4.1)$$

Now mean time to system failure (MTSF) is given by

$$E(T_0) = \lim_{s\to 0} R_0^*(S) = \frac{\mu_0 + P_{01}\mu_1 + P_{03}\mu_3}{1-2(P_{00}^{(1)} + P_{00}^{(3)})}$$

5. Expected up-time of units :

Let $A_i^a(t), A_i^b(t)$, and $A_i^c(t)$ be the respective probabilities that the units A, B and C one up (operative) at time t when initially system starts functioning from state S_i(i=0,1,3) by using definitions' of $A_i(t)$ and simple probabilistic concepts, the recurrence relations among $A_i^a(t), A_i^b(t)$, and $A_i^c(t)$ $:\bar{i} = 0,1,3$ Can easily be developed. Using technique of Laplace Transformation, the value of $A_i^a(t), A_i^b(t)$ and in terms of L.T. are as follows:

$$A_0^{a*}(s) = \frac{Z_0^*U + Z_3^*V}{U - q_{30}^*V - q_{10}^*W},$$

$$A_0^{b*}(s) = \frac{Z_0^*U + Z_1^*W}{U - q_{30}^*V - q_{10}^*W}$$

and $$A_0^{c*}(s) = \frac{Z_1^*W + VZ_3^*}{U - q_{30}^*V - q_{10}^*W}, \quad (5.1\text{-}5.3)$$

where $U = \left(1 - q_{11}^{(2)*} - q_{11}^{(4)*}\right)\left(1 - q_3^{(5)*}\right)$,

$$V = \left(1 - q_{11}^{(2)*} - q_{11}^{(4)*}\right)\left(q_{03}^*\right),$$

$$W = \left[q_{01}^*(1 - q_{33}^{(5)*}) + q_{03}^*q_{31}^{(2)*}\right]$$

The study state probabilities that the units A, B and C will be operative are respectively given by

$$A_0^a = \frac{(1-p_{35})(1-p_{12}-p_{14})\mu_0 + p_{03}(1-p_{12}-p_{14})\mu_3}{D}$$

$$A_0^b = \frac{(1-p_{35})(1-p_{12}-p_{14})\mu_0 + \left(p_{10}(1-p_{35}) + p_{03}p_{32}\right)\mu_3}{D}$$

$$A_0^c = \frac{\left(p_{01}(1-p_{35}) + p_{03}p_{32}\right)\mu_1 + p_{03}(1-p_{12}-p_{14})\mu_3}{D} \qquad (5.4\text{-}5.6)$$

where, $D = \mu_0 p_{01}(1-p_{35}) + \left((1-p_{35}) - p_{00}^{(3)}\right)\left(\mu_1 + p_{12}\mu_2 + p_{14} + \mu_4\right) + P_{03}P_{10}(\mu_3 + P_{35}\mu_5)$

Now the Expected up time of unit A during interval (0,t) is as follows.

$$\mu_{up}^a(t) = \int_0^t A_0^a(u)du$$

so that $\mu_{up}^b(t) = \int_0^t A_0^b(u).du.$ (5.7-5.8)

Similarly the expected up time of unit Band C during interval (0,t] may be obtained

6. Busy period analysis of Server :

Let $B_i^a(t)$ and $B_i^b(t)$ be the respective probabilities that the server is busy at time t in the repair of units A, B and C when initially system starts functioning from state S_i(i=0,1,3). By using definition of $B_i(t)$ and sample probabilistic concepts, the recurrence relations among $B_i^a(t)$ and $B_i^b(t) : \bar{i} = 0,1,3$ Can easily be developed. Using technique of Laplace Transformation, the value of $B_i^a(t)$ and $B_i^b(t)$ in terms of L.T. are as follows:

$$B_0^{a*}(s) = \frac{WZ_1^*}{U - q_{30}^*V - q_{10}^*W} \quad \text{and}$$

$$B_0^{b*}(s) = \frac{VZ_3^*}{U - q_{30}^*V - q_{10}^*W} \qquad (6.1\text{-}6.2)$$

The steady state probabilities that the server will busy in repair of units A and B are given respectively as follows.

$$B_0^a = \frac{[P_{01}(1-P_{35}) + P_{03}P_{32}]\mu_1}{D} \quad \text{and}$$

$$B_0^b = \frac{(1-P_{35})(1-P_{12}-P_{14})\mu_3}{D} \qquad (6.3\text{-}6.4)$$

The expected busy period of server in the repair unit A during interval (0, t] is

$$\mu_{busy}^a(t) = \int_0^t B_0^a(u)du$$

So that, $\mu_{busy}^b(s) = B_0^{a*}(s)/s$ (6.5-6.6)

Similarly, the expected busy period of a server in the repair of unit B during interval (0,t] may be obtained

7. Cast Benefit Analysis :

The net expected profit incurred in (0,t] is

$$C_0(t) = K_0\mu_{up}^{a}(t) + K_1\mu_{up}^{b}(t) + K_2\mu_{up}^{c}(t)$$

$$-K_3\mu_{busy}^{a} - K_4\mu_{busy}^{b} - k_5(t) \qquad (7.1)$$

The expected profit per unit time in steady state is given by

$$C_0 = K_0A_0^a + K_1A_0^b + K_2A_0^c - K_3B_0^a$$

$$-K_4B_0^b - K_5, \qquad (7.2)$$

where K_0, K_1, K_2 are the per unit time revenues by the system corresponding to the operation of units A, B and C respectively K_3 and K_4 are the amount spent per unit time in repairing the failed units A and B respectively and K_5 in the per unit time staking cost of repairman.

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