Ultra Scientist Vol. 24(1)A, 165-170 (2012).

On the degree of approximation to a function by $(N, p, q)_k$ (C,1) means of its Fourier Laguerre Series

*V.N. TRIPATHI, **AMIT GUPTA and ***YOGESH PATHAK

*Associate Professor in Mathematics, SBPG College, Baragaon, Varanasi (INDIA)
**Reader in Mathematics,
Sri Chhotu Ram Engineering and Technology, University Campus, Meerut (INDIA)
***Research Scholar in Mathematics,
SBPG College, Baragaon, Varanasi (INDIA)

(Acceptance Date 24th November, 2011)

Abstract

Superimposing the Nölund mean on Cesáro mean of order one, Singh and Khare[3] have studied the degree of approximation to a function by Cesáro-Nörlund mean of its Fourier Laguerre series under certain conditions.

Introduction

Here, in the present paper, we have discussed the degree of approximation to a function by $(N, p, q)_k$ (C,1) means of its Fourier Laguerre series under very general condition.

1. The Fourier Laguerre series of a Lebesgue integrable function f(x) in the right open interval [0, ∞) is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \qquad \alpha > -1, \quad (1.1)$$

where the coefficients a_n are defined by the formula

$$\Gamma(\alpha+1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-x} x^{\alpha} f(x) L_n^{(\alpha)}(x)\,dx \quad (1.2)$$

provided the integral on the right hand side exists in the sense of Lebesgue and $L_b^{(\alpha)}(x)$ is the n^{th} Laguerre polynomial of order $\alpha > -1$ defiend by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x).\, y^n = (1-y)^{-\alpha-1} \exp\left(\frac{-xy}{1-y}\right) (1.3)$$

We write

$$\phi(x) = (\Gamma(\alpha+1))^{-1} e^{-x} x^{\alpha} \{f(x) - f(0)\}. (1.4)$$

Let $\{S_n\}$ be the sequence of partial sums of a given infinite series Σu_n. Let σ_n denote the n^{th} (C,1) mean of the series Σu_n which is given by the sequence-sequence transformation[5]

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k \text{ [Titchmarsh (1939, p. 411]} (1.5)$$

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of constants, real or complex, with k > -1 and let us write

$$R_n^k = \sum_{m=0}^{n} p_{n-m}^k \; q_m^k \qquad (1.6)$$

Extending the definition due to Cass[1], we define the (N, p, q)$_k$-transform of the sequence $\{S_n\}$ of partial sums of the sereis Σu_n givn by the sequence-to-sequence transformation

$$t_n^k = \frac{1}{R_n^k} \sum_{m=0}^{n} p_{n-m}^k \, q_m^k \, S_m \qquad (1.7)$$

In particular, for $q_n = 1$, for every n, R_n^k reduces to

$$P_n^k = \sum_{m=0}^{n} p_{n-m}^k$$

and, consequently, the (N, p, q)$_k$ summability reduces to (N, p_n^k) summability defined by Cass[1]. In addition, for k = 1, it further reduces to (N, p_n) summability defined by Hardy[2].

Following (1.7), the (N, p, q)$_k$ transform of the sequence $\{\sigma_n\}$ of the (C, 1) means of the sequence $\{S_n\}$ of partial sums of the sereis Σu_n, or, simply, the (N, p, q)$_k$ (C,1) mean of $\{S_n\}$ will be defined by

$$T_n^k = \frac{1}{R_n^k} \sum_{m=0}^{n} p_{n-m}^k \, q_m^k \, \sigma_m \qquad (1.8)$$

2. Singh and Khare[3] have estimated the degree of approximation to a function by Cesáro Nörlund mean of its Fourier Laguerre series (1.1) by proving the following.

Theorem A : If $\{p_n\}$ is a non-negative and non-increasing sequence such that $P_n \to \infty$ as $n \to \infty$ and

$$\Phi(t) = \int_0^t |\phi(u)|du = 0\left(t^{\alpha+1} P_{[1/t]}\right) \qquad (2.1)$$

as $t \to 0$,

$$\int_w^n e^{u/2} u^{-\frac{u}{2}-\frac{5}{4}} |\phi(u)|du = 0\left(n^{\frac{-\alpha}{2}+\frac{1}{4}} P_n \right) \qquad (2.2)$$

as $n \to \infty$, where w is a fixed positive constant

$$\int_n^\infty e^{u/2} u^{-1/3} |\phi(u)|du < \infty \qquad (2.3)$$

then, for $-1 < \alpha < \frac{1}{2}$,

$$T_n(0) - f(0) = 0(P_n), \text{ as } n \to \infty . \qquad (2.4)$$

where $T_n(0)$ is the n^{th} Cesáro Nörlund mean of the Fourier Laguerre sereis (1.1) at the point x = 0.

Here, in the present paper, we have attempted to obtain the degree of approximation to a function by (N, p, q)$_k$ (C,1) mean of its Fourier Laguerre sereis (1.1) under very general condition by establishing the following.

Theorem: Let $\{p_n\}$ and $\{q_n\}$ be two sequences of constants and k > 1 such that p_n^k and q_n^k are non negative monotonic non-increasing and $R_n^k \to \infty$ as $n \to \infty$ together with satisfying

$$\sum_{m=0}^{n} \frac{p_{n-m}^k q_m^k}{m+1} = O\left(\frac{R_n^k}{n}\right), \text{ as } n \to \infty . \qquad (2.5)$$

Let $-1 < \alpha \le \frac{1}{2}$ and w a fixed positive constant.

If

$$\int_t^w u^{-\frac{\alpha}{2}-\frac{5}{4}} |\Phi(u)| du = 0\left[R^k_{[1/t]}\right], \text{ as } t \to 0, \tag{2.6}$$

$$\int_w^n e^{\frac{u}{2}} u^{-\frac{\alpha}{2}-\frac{5}{4}} |\Phi(u)| du = 0\left(R^k_n\right), \text{ as } n \to \infty, \tag{2.7}$$

and

$$\int_n^\infty e^{\frac{u}{2}} u^{-\alpha-\frac{7}{3}} |\Phi(u)| du = 0\left[n^{-\frac{\alpha}{2}-\frac{5}{4}}. R^k_n\right] \tag{2.8}$$

then

$$T_n(0) - f(0) = 0\left[n^{\frac{\alpha}{2}-\frac{1}{4}} R^k_n\right] \tag{2.9}$$

as $n \to \infty$, where $T_n(0)$ is the n^{th} $(N, p, q)_k$ (C,1) mean of the Fourier Laguerre series (1.1) at the point x = 0.

3. The following Lemmas are needed in order to prove our main theorem.

Lemma[4] *1* [Szegö (1967), p. 177].

If α is any arbitrary real number and c, w are fixed positive constants, then

$$L^{(\alpha)}_n(x) = \begin{cases} x^{-\frac{\alpha}{2}-\frac{1}{4}}.O\left(n^{\frac{\alpha}{2}-\frac{1}{4}}\right), & \frac{c}{n} \le n \le w \\ O(n^\alpha) & , 0 \le x \le \frac{c}{n} \end{cases}$$

Lemma 2. [Szegö (1967), p. 140].

If α is any arbitrary real number, w > 0 and 0 < η < 4, then

$$\max\left\{e^{-\frac{x}{2}} x^{\frac{\alpha}{2}+\frac{1}{4}} \left|L^{(\alpha)}_n(x)\right|\right\}$$

$$\sim \begin{cases} n^{\frac{\alpha}{2}-\frac{1}{4}}, & w \le x \le (4-\eta).\ n \\ n^{\frac{\alpha}{2}-\frac{1}{12}}, & x > n \end{cases}$$

Lemma 3. It follows from (2.6) that

$$\int_0^t |\Phi(u)| du = 0\left[t^{\frac{\alpha}{2}+\frac{5}{4}}. R^k_{[1/t]}\right].$$

Proof of Lemma 3

Let us write

$$\Phi(t) = \int_t^w u^{-\frac{\alpha}{2}-\frac{5}{4}} |\phi(u)| du$$

Then, we have

$$|\phi(t)| = -t^{\frac{\alpha}{2}+\frac{5}{4}} \Phi'(t)$$

Therefore,

$$\int_0^t |\phi(u)| du = -\int_0^t u^{\frac{\alpha}{2}+\frac{5}{4}} \Phi'(u)\, du$$

$$= -\left[u^{\frac{\alpha}{2}+\frac{5}{4}}. 0\left\{R^k_{[1/u]}\right\}\right]_0^t$$

$$+ O\left[\int_0^t u^{\frac{\alpha}{2}+\frac{1}{4}}. 0\left\{R^k_{[1/u]}\right\} du\right]$$

$$= 0\left[t^{\frac{\alpha}{2}+\frac{5}{4}} 0\left(R^k_{[1/t]}\right)\right]$$

$$+ 0\left(R^k_{[1/t]}\right) \int_0^t u^{\frac{\alpha}{2}+\frac{1}{4}}\, du$$

$$= 0\left[t^{\frac{\alpha}{2}+\frac{5}{4}} R^k_{[1]t}\right]$$

4. Proof of the theorem: It is well known that

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} \qquad (4.1)$$

Therefore, the n^{th} partial sum of the series (1.1) at the point x = 0, i.e.,

$$S_n(0) = \sum_{m=0}^{n} a_m L_m^{(\alpha)}(0)$$

$$= \left(\Gamma(\alpha+1)\right)^{-1} \int_0^\infty e^{-x} x^\alpha\, f(x) \sum_{m=0}^{n} L_m^{(\alpha)}(x)\, dx$$

$$= \left(\Gamma(\alpha+1)\right)^{-1} \int_0^\infty e^{-x} x^\alpha f(x)\, L_n^{(\alpha+1)}(x)\, dx \qquad (4.2)$$

so that

$$S_n(0) - f(0) = \int_0^\infty \phi(x) L_n^{(\alpha+1)}(x)\, dx \qquad (4.3)$$

The $(N,p,q)_k (C,1)$ means of the series (1.1) at the point x = 0 will be given by following (1.8) as

$$T_n(0) - f(0) = \frac{1}{R_n^k} \sum_{m=0}^{n} \frac{p_{n-m}^k\ q_m^k}{m+1} \int_0^\infty \phi(u)\ L_m^{\alpha+2}(u) du$$

where

$$\sigma_m(0) - f(0)$$

$$= \frac{1}{m+1} \sum_{\nu=0}^{m} \left\{ s_\nu(0) - f(0)\right\}$$

$$= \frac{1}{m+1} \int_0^\infty \phi(u) \sum_{\nu=0}^{m} L_\nu^{(\alpha+1)}(u) du$$

$$= \frac{1}{m+1} \int_0^\infty \phi(u)\ L_m^{(\alpha+2)}(u) du \qquad (4.5)$$

Now, in order to prove the required result in (2.9), we write

$$T_n(0) - f(0) = \frac{1}{R_n^k} \sum_{m=0}^{n} \frac{p_{n-m}^k q_m^k}{m+1} \left\{ \int_0^{c/n} + \int_{c/n}^{w} + \int_w^n + \int_n^\infty \right\}$$

$$\times\ \phi(n)\, L_n^{(\alpha+2)}(u) du$$

$$= I_1 + I_2 + I_3 + I_4, \quad \text{say} \qquad (4.6)$$

We are now required to show that

$$I_j = 0\left[n^{\frac{\alpha}{2}-\frac{1}{4}} . R_n^k \right] \qquad (4.7)$$

as $n \to \infty$, for j = 1,2,3,4.

Let us first consider I_1. Now,

$$I_1 = O\left(\frac{1}{R_n^k}\right) . \sum_{m=0}^{n} \frac{p_{n-m}^k\ q_m^k}{m+1} . m^{\alpha+2} \int_0^{c/n} |\phi(u)| du$$

$$= O\left(\frac{1}{R_n^k}\right) . O\left(\frac{R_n^k}{n} . n^{\alpha+2}\right) . \ o\left(n^{-\frac{\alpha}{2}-\frac{5}{4}} R_m^k\right)$$

$$= 0\left[n^{\frac{\alpha}{2}-\frac{1}{4}} R_n^k \right] \qquad (4.8)$$

as $n \to \infty$, using Lemma 1, condition (2.5) and Lemma 3, respectively.

Next, considering I_2 and using again Lemma 1, conditions (2.5) and (2.6), we have

$$I_2 = O\left(\frac{1}{R_n^k}\right) . \sum_{m=0}^{n} \frac{p_{n-m}^k \; q_m^k}{m+1} . O\left(m^{\frac{\alpha}{2}+\frac{3}{4}}\right)$$

$$\times \int_{c/w}^{\infty} u^{-\frac{\alpha}{2}-\frac{5}{4}} |\phi(u)| du$$

$$= O\left(\frac{1}{R_n^k} . \frac{R_n^k}{n} \; n^{\frac{\alpha}{2}+\frac{3}{4}}\right) . 0\left(R_n^k\right)$$

$$= 0\left(n^{\frac{\alpha}{2}-\frac{1}{4}} R_n^k\right) \text{ as } n \to \infty. \qquad (4.9)$$

Further, we consider I_3. Using Lemma 2 and conditions (2.5) and (2.7), we have

$$I_3 = O\left(\frac{1}{R_n^k}\right) . \sum_{m=0}^{n} \frac{p_{n-m}^k \quad q_m^k}{m+1} \; . \; \int_{\omega}^{n} e^{\frac{u}{2}} u^{-\frac{\alpha}{2}-\frac{5}{4}} . n^{\frac{\alpha}{2}+\frac{3}{4}} |\phi(u)| du$$

$$= O\left(\frac{1}{R_n^k} . \frac{R_n^k}{n} . n^{\frac{\alpha}{2}+\frac{3}{4}}\right) . \int_{\omega}^{n} e^{\frac{u}{2}} u^{-\frac{\alpha}{2}-\frac{s}{4}} |\phi(u)| du$$

$$= O\left(n^{\frac{\alpha}{2}-\frac{1}{4}}\right) . 0\left(R_n^k\right)$$

$$= 0\left(n^{\frac{\alpha}{2}-\frac{1}{4}} . R_n^k\right), \text{ as } n \to \infty \qquad (4.10)$$

Lastly, we consider I_4. Using Lemma 2 and conditions (2.5), (2.8), we have

$$I_4 = O\left(\frac{1}{R_n^k}\right) \sum_{m=0}^{n} \frac{p_{n-m}^k \; q_m^k}{m+1} \int_{n}^{\infty} e^{\frac{u}{2}} u^{-\frac{\alpha}{3}-\frac{5}{4}} |\phi(u)|$$

$$\times n^{\frac{\alpha}{2}+\frac{11}{12}} du$$

$$= O\left(\frac{1}{R_n^k} . \frac{R_n^k}{n} \; . \; n^{\frac{\alpha}{2}+\frac{11}{12}}\right) \; . \; \int_{n}^{\infty} e^{\frac{u}{2}} \; u^{-\frac{\alpha}{2}-\frac{5}{4}} |\phi(u)| du$$

$$= O\left(n^{\frac{\alpha}{2}-\frac{1}{12}}\right) . \int_{n}^{\infty} e^{\frac{u}{2}} \; u^{-\frac{\alpha}{2}-\frac{5}{4}} |\phi(u)| du$$

$$= O\left(n^{\frac{\alpha}{2}-\frac{1}{12}}\right).\int_n^\infty e^{\frac{u}{2}}\, u^{-\alpha-\frac{7}{3}}.u^{\frac{\alpha}{2}+\frac{13}{12}}\,|\phi(u)|\,du$$

$$= O\left(n^{\frac{\alpha}{2}-\frac{1}{12}}.n^{\frac{\alpha}{2}+\frac{13}{12}}\right).\int_n^\infty e^{\frac{u}{2}}\, u^{-\alpha-\frac{7}{3}}\,|\phi(u)|\,du$$

$$= O\left(n^{\alpha+1}\right).\int_n^\infty e^{\frac{u}{2}}\, u^{-\alpha-\frac{7}{3}}\,|\phi(u)|\,du$$

$$= O\left(n^{\alpha+1}\right).o\left(n^{-\frac{\alpha}{2}-\frac{5}{4}}.R_n^k\right)$$

$$= O\left(n^{\frac{\alpha}{2}-\frac{1}{4}}.R_n^k\right),\ \text{as}\ n\to\infty \qquad (4.11)$$

Collecting (4.8), (4.9), (4.10) and (4.11), we get the required result in (4.7). This completes the proof of our theorem.

References

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