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A study of (ε)- Almost Paracontact Metric Manifold-1

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Abstract

The object of the present paper is to studies of special $(\in)$– para Sasakian manifold and its properties. We have shown that a recurrent special $(\in)$– para Sasakian manifold with η as 1-form of recurrence is Einstein manifold. Among others,

Key words: Indefinite almost paracontact metric manifold, $(\in)$– para Sasakian manifold, 2-Killing vector fields.

Introduction

Takahashi[13] introduced almost contact manifold equipped with associated pseudo-Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifold and indefinite Sasakian manifolds are also known as $(\in)$ – almost contact metric manifolds and $(\in)$– Sasakian manifolds, respectively[1,6,7]. The notion of almost paracontact manifold was introduced by Sato[4] in 1976. The structure is an analogue of the almost contact structure[3,12] and is closely related to almost product structure [in contrast to almost contact structure, which is related to almost complex structure]. An almost contact manifold is always odd dimensional but an almost paracontact manifold could be even dimensional as well. In 1969, Also, in 1989, Matsumoto[8] replaced the structure vector field ξ by - ξ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field ξ is always lime like. Motivated by these circumstances M.M. Tripathi *et. al.*[9] associated a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite almost paracontact metric structure an $(\in)$– almost paracontact structure, where the structure vector field ξ is space like or time like according as $\in=1$ or $\in=-1$. In[10] the authors studied $(\in)$– almost para contact manifolds, and in particular $(\in)$– para Sasakian manifold. They gave basic definitions and some examples of $(\in)$– almost paracontact manifolds and introduced the notion of an $(\in)$– para Sasakian structure. The basic properties, some typical, identities for curvature tensor and Ricci tensor of $(\in)$– para Sasakian manifold were also studied in[9]. The authors proved that if a semi-Riemannian manifold is one of flat proper recurrent or proper Ricci-Recurrent than it cannot admit an $(\in)$– para Sasakian structure. Also, they showed that, for an $(\in)$ – para Sasakian manifold, the conditions of being symmetric, semi-symmetric or of constant sectional curvature are all identical.

In subsequent paper, M.M. Tripathi[9,10] *et.al.* studied 3 – dimensional $(\in)$– para Sasakian manifolds. They obtained a necessary and sufficient condition for an $(\in)$– para Sasakian 3 – manifolds to be an indefinite space form.

The present paper is the continuation of previous studies. In section 2, we sketch the notion of $(\in)$– almost paracontact metric manifold, $(\in)$ – para Sasakian manifold and its properties. In section-3, we introduce special $(\in)$– para Sasakian manifold and prove that it is an $(\in)$– para Sasakian manifold. Section-4, deals with recurrence properties of special $(\in)$– para Sasakian manifold.

2 – Preliminaries :

Let M be an n-dimensional almost paracontact manifold[4] equipped with an almost paracontact structure (φ, ξ, η) consisting of a tensor field of φ type (1, 1), a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi(\xi) = 0, \tag{2.3}$$

and

$$\eta 0 \varphi = 0. \tag{2.4}$$

Throughout this paper, we assume that $X, Y, Z, U, V, W \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields in M, unless specifically stated otherwise. By a semi-Riemannian metric[2] on a manifold M, we understand a non-degenerate symmetric tensor field g of type (0, 2). In particular, if its index is 1, it becomes a Lorentzian metric[5]. Let g be a semi-Riemannian metric with index (g) = v in an n-dimensional almost paracontact manifold M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

where $\in= \pm 1$. Then M is called an $(\in)$– almost paracontact metric manifold equipped with an $(\in)$– almost paracontact metric structure[9] $(\varphi, \xi, \eta, g, \varepsilon)$. In particular, if index $(g) = 1$, then an $(\in)$–almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. In particular, if the metric is positive definite, then an $(\in)$– almost paracontact metric manifold is the usual almost paracontact metric manifold[14].

The condition (2.5) is equivalent to

$$g(X, \varphi Y) = g(\varphi X, Y) \tag{2.6}$$

along with

$$g(X, \xi) = \in \eta(X). \tag{2.7}$$

From (2.7), if follows that

$$g(\xi, \xi) = \in. \tag{2.8}$$

that is, structure vector field ξ is never light like. Defining

$$\phi(X, Y) = g(X, \varphi Y), \tag{2.9}$$

we note that

$$\phi(X, \xi) = 0. \tag{2.10}$$

From (2.9), we also have

$$(\nabla_X \phi)(Y, Z) = g(\nabla_X \varphi)(Y, Z) = (\nabla_X \phi)(Z, Y). \tag{2.11}$$

If on an $(\in)$– almost paracontact metric manifold M

$$2\phi(X, Y) = (\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) \tag{2.12}$$

for all $X, Y \in TM$, then M is called as $(\in)$ – paracontact metric manifold[11].

An $(\in)$– almost paracontact metric structure $(\varphi, \xi, \eta, g, \in)$ is called an $(\in)$–s– paracontact metric structure if

$$\nabla \xi = \in \varphi. \tag{2.13}$$

A manifold equipped with an $(\in)$ – S – paracontact metric structure is called an $(\in)$– S – paracontact metric manifold.

Equation (2.13) is equivalent to

$$\phi(X, Y) = g(\varphi X, Y) = \in g(\nabla_X \xi, Y) = (\nabla_X \eta)(Y) \tag{2.14}$$

for all $X, Y \in TM$. An $(\in)$–almost paracontact metric structure is called an $(\in)$–para Sasakian structure if [9]

$$(\nabla_X \varphi)(Y) = g(\varphi X, \varphi Y)\xi - \in \eta(Y)\varphi^2 X, \tag{2.14}$$

where Δ is the Levi – Civita connection with respect to g. A manifold endowed with an $(\in)$– para Sasakian structure is called an $(\in)$– para Sasakian manifold.

For $\in= 1$ and g Riemannian, M is the usual para–Sasakian manifold[4]. For, $\in= -1$, g Lorentzian and ξ replaced by -ξ, M becomes a Lorentzian Para Sasakian manifold[8].

It is known that on an $(\in)$– para Sasakian manifold[9]

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \tag{2.16}$$

$$R(X, Y, Z, \xi) = -\eta(X)g(Y, Z) + \eta(Y)g(X, Z), \tag{2.17}$$

$$\eta(R(X, Y, Z)) = - \in \eta(X)g(Y, Z) + \in \eta(Y)g(Y, Z), \tag{2.18}$$

$$R(\xi, X, Y) = - \in g(X, Y)\xi + \eta(Y)X, \tag{2.19}$$

and

$$S(X, \xi) = -(n-1)\eta(X). \tag{2.20}$$

3 – Special $(\in)$ *– Para Sasakian Manifold:*

Consider an n-dimensional manifold admitting a 1 – form η, a vector field ξ and a pseudo – Riemannian metric *g* satisfying

$$(\nabla_X \eta)(Y) = -\in g(X,Y) + \eta(X)\eta(Y) \quad (3.1)$$

and $g(X,\xi) = \in \eta(X)$. (3.2)

Further, if we put

$$\nabla_X \xi = \in \varphi(X), \quad (3.3)$$

where φ is tensor field of type (1,1), then

$$\in g(\varphi X, Y) = g(\nabla_X \xi, Y). \quad (3.4)$$

The equation (3.2), yields

$$(\nabla_X \eta)(Y) = \in g(\nabla_X \xi, Y), \quad (3.5)$$

which, together with (3.4), implies

$$\phi(X,Y) = g(\varphi X, Y) = (\nabla_X \eta)(Y). \quad (3.6)$$

Form (3.6) and (3.1), we get

$$\bar{X} = \in [-X + \eta(X)\xi], \quad (3.7)$$

which produces easily

$$\bar{\bar{X}} = X - \eta(X)\xi. \quad (3.8)$$

Also, (3.7) yields

$$\eta(\xi) = 0 \quad (3.9)$$

and $g(\bar{X},\bar{Y}) = g(X,Y) - \varepsilon\eta(X)\eta(Y)$. (3.10)

Differentiating (3.7) covariantly, we obtain

$$(\nabla_X \varphi)(Y) = \varepsilon[(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi],$$

which, in view of (3.3) and (3.1) reduces to

$$(\nabla_X \varphi)(Y) = -g(\bar{X},\bar{Y})\xi - \varepsilon\eta(Y)\varphi^2 X. \quad (3.11)$$

All above results prove that the manifold satisfying (3.1), (3.2) and (3.3) is an $(\in)$– para Sasakian manifold. We call this as special $(\in)$– para Sasakian manifold.

Theorem 3.1: On special $(\in)$ – para Sasakian manifold, the 1 – form η - is closed.

Proof: The proof of the theorem is obvious.

Theorem 3.2: On special $(\in)$ – para Sasakian manifold M, the vector field ξ is 2-killing vector field.

Proof: A vector field is said to be 2-Killing if it satisfies[15]

$$R(\xi, X, \xi, X) = g(\nabla_X \nabla_\xi \xi, X) + g(\nabla_X \xi, \nabla_X \xi). \quad (3.12)$$

From (2.17), we get

$$R(\xi, X, \xi, X) = g(\bar{X}, \bar{X}). \quad (3.13)$$

Again, we have

$$g(\nabla_X \nabla_\xi \xi, X) + g(\nabla_X \xi, \nabla_X \xi) = g(\varphi X, \varphi X). \quad (3.14)$$

From (3.8) and (3.9), we get

$$R(\xi, X, \xi, X) = g(\nabla_X \nabla_\xi \xi, X) + g(\nabla_X \xi, \nabla_X \xi),$$

which implies that ξ is 2-killilng vector field.

4 – Recurrent Special $(\in)$– Para Sasakian Manifold :

In this section, we consider a special $(\in)$– Para Sasakian Manifold, which satisfies

$$(\nabla_X R)(Y,Z,U) = \eta(X) R(Y,Z,U). \quad (4.1)$$

Theorem 4.1: If a special $(\in)$– para Sasakian manifold satisfies (4.1) then.

$$R(X,Y,Z) = [g(Z,Y) - \eta(Y)\eta(Z)]X$$

$$+\left[g(Z,X)+\eta(X)\eta(Z)\right]Y.$$

Proof: Let special $(\in)$–para Sasakian manifold *M* satisfies

$$(\nabla_X R)(Y,Z,W)=\eta(X)R(Y,Z,W).$$

Now, putting $W=\xi$ in (4.1), we get

$$(\nabla_X R)(Y,Z,\xi)=\eta(X)R(Y,Z,\xi). \tag{4.3}$$

Using (2.16) in above, we get

$$(\nabla_X R)(Y,Z,\xi)=\left[\eta(Y)Z-\eta(Z)Y\right]\eta(X). \tag{4.4}$$

Differentiating (2.16) covariantly, we get

$$(\nabla_X R)(Y,Z,\xi)+R(Y,Z,\varphi X)$$
$$=(\nabla_X\eta)(Y)Z-(\nabla_X\eta)(Z)Y. \tag{4.5}$$

Using (4.4) in (4.5), we get

$$R(Y,Z,\varphi X)=(\nabla_X\eta)(Y)Z-(\nabla_X\eta)(Z)Y$$
$$-\eta(X)(\eta(Y)Z-\eta(Z)Y),$$

which in view of (3.1), gives

$$R(Y,Z,\varphi X)=-\in g(X,Y)Z-\in g(X,Z)Y. \tag{4.6}$$

Using (3.7) in above, we get

$$R(X,Y,Z)=\left[g(Y,Z)-\eta(Y)\eta(Z)\right]X$$
$$+\left[g(X,Z)+\eta(X)\eta(Z)\right]Y. \tag{4.7}$$

Theorem 4.2: If special $(\in)$ – para Sasakian manifold satisfies (4.1) then it is η - Einstein manifold with constant associated scalar.

Proof: An $(\in)$– paracontact manifold is called η–Einstein if its Ricci tensor satisfies

$$S(Y,Z)=ag(Y,Z)-b\eta(Y)\eta(Z)$$

for some smooth functions *a* and *b*.
Now, contracting (4.7), we get

$$S(Y,Z)=(n+1)g(Y,Z)-(n-1)\eta(Y)\eta(Z). \tag{4.8}$$

This proves the result.

Corollary 4.1: Special $(\in)$ – para Sasakian manifold satisfying (4.1) is a manifold of constant scalar curvature.

Proof: From (4.8), we have

$$r=n(n+1)-\varepsilon(n-1). \tag{4.9}$$

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