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Remarks on rg-closure, gpr-closure operators and gpr-separation axioms

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Abstract

Recently Jin Han Park[8,9] *et.al.*[**Further results on rg-continuity, Far *East J. Math.Soc.2000 special volume part* 11, 237-244 and On preserving rg-closed sets, *East Asian Math.J.*16(1) (2000),125-133**] studied rg-continuity and some preservation theorems using rg-closure operator and Gnanambal *et.al.*[**On gpr-continuous functions in topological spaces, *Indian J.Pure.appl.Math.*,30(6)(1999), 581-593**] discussed gpr-continuous functions using gpr-closure operator. Quite Recently Balasubramanian[4] *et.al.*[**gpr-separation axioms-I, *IJMA*, 2(10) (2011), 2055-2067**] discussed several types of gpr-separation axioms. The purpose of this paper is to investigate some properties of rg-closure operator, gpr-closure operator and gpr-separation axioms. The most important result that is proved in this paper is " In a topological space every singleton set is rg-open" that leads to the investigation of some of the results due to Jin Han Park et. al. Gnanambal et. al. and separation axioms due to Balasubramanian *et. al.*

Keywords and Phrases: rg-open, rg-closed, gpr-open, gpr-closed etc.,

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1. Introduction

In 1993, Palaniappan and Chandrasekhara Rao[13] defined the concept of regular-generalized closed sets. Arockiarani and Balachandran[2] introduced the notion of rg-closure operator. Recently Jin Han Park *et.al.*[8,9] studied rg-continuity and some preservation theorems using rg-closure operator and Gnanambal *et.al.*[6,7] discussed gpr-continuous functions using gpr-closure operator. Quite Recently Balasubramanian *et.al.*[3,4] discussed several types of gpr-separation axioms. The purpose of this paper is to investigate some properties of rg-closure operator, gpr-closure operator and gpr-separation axioms. The most important result that is proved in this paper is " In a topological space every singleton set is rg-open" that leads to the investigation of some of the results due to Jin Han Park *et. al.*[8,9], Gnanambal *et. al.*[6,7] and separation axioms due to Balasubramanian *et. al.*[3,4].

2. Preliminaries :

Throughout this paper (X,τ) denotes a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, *cl*(A) and *int*(A) respectively denote the closure of A and the interior of A in (X,τ). X\A denotes the complement of A in X and $\wp$(X) denotes the power set of X. We recall the following definitions and results that will be useful. A subset A of a topological space (X,τ) is regular open if A= *int*(*cl*(A))[14], regular closed if A= *cl*(*int*(A)), pre-open if A ⊆ *int*(*cl*(A))[12] and pre-closed if *cl*(*int*(A)) ⊆ A. Again a subset B of a topological space (X, τ) is called generalized closed[10] (briefly g-closed) if *cl*(B)⊆U whenever B⊆U and U is open in X and regular generalized closed (briefly rg-closed)[13] if *cl*(B)⊆U whenever B⊆U and U is regular open in X. The pre-closure of a subset A of X is the intersection of all pre-closed sets containing A and is denoted by *pcl*(A). The intersection of all g-closed (resp. rg-closed) sets containing B is called the g-closure (resp. rg-closure) of B and denoted[2] by $c^*(B)$ [11] (resp. $cl_r^*(B)$). The complement of a rg-closed set is rg-open.

Lemma 2.1: A subset A of X is rg-open if and only if F⊆ *int*(A) whenever F⊆ A, F is regular closed.[13]

Definition 2.2: Let (X,τ) be a topological space and B⊆X. Then B is called

(i) generalized pre-closed(briefly gp-closed)[11] if *pcl*(B)⊆U whenever B⊆U and U is open in X.

(ii) generalized pre-regular closed(briefly gpr-closed)[6] if *pcl*(B)⊆U whenever B⊆U and U is regular open in X.

The complement of a gp-closed set is gp-open and that of a gpr-closed set is gpr-open. The intersection of all gp-closed (resp. gpr-closed) sets containing B is called the gp-closure (resp. gpr-closure) of B and denoted by $pcl^*(B)$ [11] (resp. *gpr-cl*(B)[7]). In this paper, *PO*(X,τ) (resp. *RGO*(X,τ), resp. *RGC*(X,τ), resp. *GPRO*(X,τ), resp. *GPRC*(X,τ)) denotes the class of all pre-open(resp. rg-open, resp. rg-closed, resp. gpr-open, resp. gpr-closed) sets in (X,τ).

Definition 2.3: For a subset A of X, *gpr-int*(A)[7] is defined as the union of all gpr-open sets contained in A.

Definition 2.4: A space (X,τ) is called $T_{1/2}$ [10] (resp. Pre-regular[6]) if every g-closed (resp. gpr-closed) set is closed(resp. pre-closed).

Definition 2.5: Let (X,τ) be a topological[7,8,11] space. Then

(i) τ_g^* ={V⊆X: *gpr-cl*(X\V)=X\V}.

(ii) τ_r^* ={V⊆X: $cl_r^*(X \setminus V)$ =X\V}.

(iii) ${}_{p}\tau^{*}$ = {V⊆X: $pcl^{*}(X\setminus V)$ =X\V}.

(iv) τ_η ={A⊆X : A∩B∈*GPRO*(X,τ) for all[7] B∈*GPRO*(X,τ)}.

Remark 2.6: Let (X,τ) be a topological space. Since closed⇒g-closed⇒rg-closed⇒ gpr-closed, we have A⊆*gpr-cl*(A)⊆ $cl_r^*(A)$ ⊆ $c^*(A)$ ⊆*cl*(A) for every subset A of X.

3. *rg-closure and gpr-closure :*

In this section, the operators namely rg-closure and gpr-closure are characterized. The next theorem is frequently used in this paper.

Theorem 3.1: In a topological space (X,τ), the following hold:

(i) {x} is rg-open for every x∈X.

(ii) $cl_r^*(A)$ = *gpr-cl*(A) = A, for every subset A of X.

Proof: Let (X,τ) be a topological space. Fix x∈X. Suppose A⊆{x} and A is regular closed. In order to prove that {x} is rg-open, it is enough to prove that A⊆*int* ({x}). This is true if {x} is open. Suppose {x} is not open. Then *int*({x})=∅.

Since A is regular closed, A= *cl*(*int*(A)) ⊆ *cl*(*int*({x}))=∅ that implies A=∅. This shows that A=∅⊆ *int*({x}). Then by Lemma 2.1, it follows that{x} is rg-open. This proves (i).

For any subset A of X, the inclusion A⊆ $cl_r^*(A)$ is always true. To establish the reverse inclusion we take x∈ $cl_r^*(A)$. Suppose x∉A. Then A⊆X\{x} and x∉X\{x}. By using (i), {x} is rg-open. Since X\{x} is rg-closed and A⊆X\{x}, we have $cl_r^*(A)$ ⊆X\{x}. This implies that x∉ $cl_r^*(A)$ contradicting x∈ $cl_r^*(\{A\})$. This shows that x∈A. Therefore A= $cl_r^*(A)$. By applying this and using Remark 2.6, we get *gpr-cl*(A)=A. This establishes (ii).

Remark 3.2: For a subset A of a topological space (X,τ), if $cl_r^*(A)$ =A, then A need not be rg-closed and if *gpr-cl*(A)=A, then A need not be gpr-closed as shown in the next examples.

Example 3.3: Let X={a, b, c, d}, τ={∅,{a},{b},{a, b},{a, b, c}, X}. Here *RGC*(X)={∅,{c},{d}, {a, b }, {a,c},{a,d}, {b,c}, {b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},X}. Now $cl_r^*(\{a\})$ = {a}, which is not rg-closed.

Gnanambal[7] [Example 3.5] established that if *gpr-cl*(A)=A, then A need not be gpr-closed. However the following example also shows this.

Example 3.4: Let X ={a, b, c, d }, τ={∅,{a},{b},{a, b},{b, c},{a, b, c}, X}. Here *GPRC*(X)={∅,{c},{d}, {a, b }, {a, c},{a, d}, {b, d},{c, d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},X}. Now *gpr-cl*({a})={a}, which is not gpr-closed.

Jin Han Park[9] *et.al.* [Theorem 2.12] established the equivalence of the following statements in any topological space (X,τ).

(i) For each x∈X, X\{x} is rg-closed in X.

(ii) If {x} is regular closed in X, then {x} is regular open in X.

However the above statements are always true in any topological space as shown in the next proposition.

Proposition 3.5. Let (X,τ) be a topological space. Then the following hold:

(i) For each x∈X, X\{x} is rg-closed in X.
(ii) If {x} is regular closed in X, then {x} is regular open in X.

Proof: Let (X,τ) be a topological space. (i) Follows from Theorem 3.1. Suppose {x} is regular closed. By using Theorem 3.1, {x} is rg-open in X. Since {x} is rg- open, by Lemma 2.3, {x}⊆*int*({x}) that implies {x}=*int*({x}). Since {x} is regular closed, {x}=*int*({x}) =*int*(*cl*({x}) that implies{x} is regular open in X.

Gnanambal[7] [Lemma 2.1] established that for every x∈X, X\{x} is gpr-closed or regular open. However X\{x} is always gpr-closed as shown in the next corollary.

Corollary 3.6: For every x∈X, X\{x} is gpr-closed. Consequently {x} is gpr-open.

Proof: Let (X,τ) be a topological space and let x∈X. By using Theorem 3.1, {x} is rg-open. This implies X\{x} is rg-closed in X. Since every rg-closed set is gpr-closed, X\{x} is gpr-closed. Since X\{x} is gpr-closed, {x} is gpr-open.

Theorem 3.7: Let (X,τ) be a topological space. Then $\tau_r^* = \tau_g^* = \wp(X)$.

Proof: Follows from Definition 2.5 and Theorem 3.1.

The following definition is due to Chawalit Boonpok[5]

Definition 3.8: A topological space X is said to be rg-Hausdorff [5] [Definition 8] if whenever x and y are distinct points of X there are disjoint rg-open sets U and V with x∈U and y∈V.

Remark 3.9: Since {x} is rg-open for every x in X, every topological space is rg-Hausdorff. Consequently the preservation theorems[5] [Theorem 3, Theorem 4] of Chawalit Boonpok are obvious.

Remark 3.10: The proof for Theorem 3.2 of [9] follows easily from the fact $cl_r^*(A)$ = A. Also Gnanambal[7] [Theorem 3.14] established that τ_g^* is a topology provided *GPRO*(X,τ) is a topology. But the condition "*GPRO*(X,τ) is a topology" is not necessary because $\tau_g^* = \wp(X)$.

Lemma 3.11: Let (X,τ) be a topological space. Let A⊆X and $\{A_\alpha\}$ be a collection of subsets of (X,τ). Then

(i) $gpr-cl(\bigcup_\alpha A_\alpha) = \bigcup_\alpha gpr-cl(A_\alpha)$.

(ii) $cl_r^*(\bigcup_\alpha A_\alpha) = \bigcup_\alpha cl_r^*(A_\alpha)$.

(iii) $gpr-cl(\bigcap_\alpha A_\alpha) = \bigcap_\alpha gpr-cl(A_\alpha)$.

(iv) $cl_r^*(\bigcap_\alpha A_\alpha) = \bigcap_\alpha cl_r^*(A_\alpha)$

(v) *gpr-cl*(X\A)= X*gpr-cl*(A).

(vi) $cl_r^*(X \setminus A) = X \setminus cl_r^*(A)$.

Proof: Follows from the fact that, *gpr-cl*(A)=A and $cl_r^*(A)$ =A.

Remark 3.12: Anitha[1] *et. al*. [Lemma 4.5], proved that, *gpr-cl*(A∪B)=*gpr-cl*(A)∪*gpr-cl*(B) if *gpr-cl*(A) is gpr-closed and semi-closed and if *gpr-cl*(B) is gpr-closed. But the conditions are not necessary as was shown in Lemma 3.11.

The following Theorem is an improved version of Theorem 3.15 of Gnanambal[7].

Theorem 3.13: Let (X,τ) be a topological space. Then the following hold:

(i) (X,τ) is pre-regular $T_{1/2}$ if and only if *PO*(X,τ) = ℘(X).

(ii) Every gpr-closed set is closed if and only if τ is discrete.

(iii) If every gpr-closed set is gp-closed, then $_p\tau^* = \wp(X)$.

Proof. Follows from Theorem[7] 3.15 and Theorem 3.7.

Theorem 3.14: If *PO*(X,τ) is a topology, then *GPRO*(X,τ)= $\tau_\eta = \tau_r^* = \tau_g^* = \wp(X)$.

Proof: Follows from Proposition 3.28 of [7] and Theorem 3.7.

Proposition 3.15: (X, τ_r^*) is discrete.

Proof: Follows from Theorem 3.14.

Corollary[9] 3.16: (Theorem 2.10). (X, τ_r^*) is a $T_{1/2}$ space.

*Proof:*Follows from Proposition 3.15.

Corollary[8] 3.17: (Corollary 2.11). $(\tau_r^*)^* = (\tau_r^*)_r^* = \tau_r^*$.

Proof: Follows from Proposition 3.15.

Theorem 3.18: In a topological space, *gpr-int*(A)=A, for every A⊆X.

Proof. Let (X,τ) be a topological space. By Lemma 3.27 of [7], X \ *gpr-int*(A) = *gpr-cl*(X\A). By Theorem 3.1, *gpr-cl*(A)=A that implies X \ *gpr-int*(A)= *gpr-cl*(X\A)=X\A which further implies *gpr-int*(A)=A.

From the fact that $cl_r^*(A)$=A, Theorem 4.3 of [7] is obvious. The following is the simplified version of Theorem 4.5 of [7].

Theorem 3.19: For any map f: (X,τ)→(Y,σ), the map f: $(X, \tau_g^*) \to (Y,\sigma)$ is continuous.

Proof: Follows from the fact $\tau_g^* = \wp(X)$.

4. *gpr-separation axioms :*

Balasubramanian and Lakshmi Sarada[3,4] introduced a variety of separation axioms, namely gpr-R_0, gpr-R_1, gpr-T_1 and gpr-T_2. They are further investigated in this section.

Definition[4] *4.1:* A topological space (X,τ) is said to be gpr-T_1 (resp. gpr-T_2) [Definition 2.3] if for any two distinct points x and y of X, there exist (resp. disjoint) gpr-open sets U and V such that $x\in U$ and $y\in V$.

Definition[3] *4.2:* A topological space (X,τ) is said to be gpr-R_0 space if every gpr-open set contains the gpr-closure of each of its points.

Definition[4] *4.3:* A topological space (X,τ) is said to be gpr-R_1.[Definition 2.3] if and only if for any x, $y\in X$ with *gpr-cl*({x}) ≠*gpr-cl*({y}), there exist disjoint gpr-open sets U and V such that *gpr-cl*({x})$\subseteq$U and *gpr-cl*({y})$\subseteq$V.

Definition 4.4: Let (X,τ) be a topological space. A point x is said to be a gpr-limit (resp. gpr-T_0 limit) [Definition 3.01] point of A if each gpr-open set containing x contains some point y of A such that $x\neq y$ (resp. *gpr-cl*({x}) ≠*gpr-cl*({y})).

Proposition 4.5: Let (X,τ) be a topological space. Then the following hold.

(i) (X,τ) is gpr-R_0, (ii) (X,τ) is gpr-R_1, (iii) (X,τ) is gpr-T_1 and (iv) (X,τ) is gpr-T_2.

Proof: Follows from Theorem 3.1(ii), Definition 4.1, Definition 4.2 and Definition 4.3.

Since *gpr-cl*({x}) ={x} and *gpr-cl*({y})={y}, it follows that x is a gpr-T_0-limit point of A if and only if x is a gpr-limit point of A where $x,y\in X$ and $A\subseteq X$. It has been observed that gpr-T_0 identification space introduced by to Balasubramanian *et. al*.[4], is nothing but discrete. More precisely, the equivalence relation given in Definition 5.01 of Balasubramanian *et. al*. is nothing but equality relation on a topological space (X,τ). According to Balasubramanian *et. al*.[4], $x\,\Re\,y$ if and only if *gpr-cl*({x})=*gpr-cl*({y})) that is if and only if x=y for every $x,y\in X$. Consequently $X_0=X$ and $Q(X_0)=\wp(X)$. This establishes that $(X_0, Q(X_0))$ is discrete[10].

The following Definition is due to Balasubramanian *et. al*.[4].

Definition 4.7: A function $f : X\rightarrow Y$ is said to be almost-gpr-irresolute[3] [Definition 7.2] if for each x in X and each gpr-neighborhood V of *f*(x), *gpr-cl*(f^{-1}(V)) is a gpr-neighborhood of x .

Remark 4.8: Since *gpr-cl*(f^{-1}(V))=f^{-1}(V), the gpr-irresolute function of Gnanambal[6] and almost gpr-irresolute function of Balasubramanian *et. al*.[4] represent the same concept.

5. Conclusion

In this paper the following results are established.

(i) Every singleton is rg-open and hence every singleton is gpr-open.
(ii) Every topological space is rg-Hausdorff.
(iii) Every topological space is gpr-R_i for i=0,1.
(iv) Every topological space is gpr-T_i for i=1,2.

References

1. M. Anitha and P. Thangavelu, A note on

gpr-closed sets in topology, *Journal of Indian Acad. Math. 28(2),* 475-481 (2006).
2. I. Arockiarani and K. Balachandran, On regular generalized continuous maps in topological spaces, *Kyungpook Math. J. 37,* 305-314 (1997).
3. S. Balasubramanian and M. Lakshmi Sarada, GPR-separation axioms, *Bull. Kerala Math. Association, 8(3)*, 157-173 (2011).
4. S. Balasubramanian and M. Lakshmi Sarada, GPR-separation axioms-I, *IJMA, 2(10),* 2055-2067 (2011).
5 Chawalit Boonpok, Preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces, *Naresuan University Journal 11(3),* 75-77 (2003).
6. Y. Gnanambal, On generalized pre regular closed sets in topological spaces, *Indian J. Pure Appl. Math.* 28(3), 351-360 (1997).
7. Y. Gnanambal and K. Balachandran, On gpr-continuous functions in topological spaces, *Indian J. Pure. Appl. Math. 30(6),* 581-593 (1999).
8. Jin Han Park, Jin Keun Park and Seong Jun Park, Further results on rg-continuity, *Far East J. Math.Soc. special volume part 11,* 237-245 (2000).
9. Jin Han Park, Jin Keun Park and Seong Jun Park, On preserving rg-closed sets, *East Asian Math. J. 16(1)*,125-133 (2000).
10. N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat.Palermo 9(2),* 89-96 (1970).
11. Maki, J. Umehara, T.Noiri, Every topological space is pre-$T_{1/2}$, *Mem. Fac. Sci. Kochi, Univ. (Math) 17,* 33-42 (1996).
12. A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On Precontinuous and Weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt 53,* 47-53 (1982).
13. N. Palaniappan and K.C. Rao, Regular generalized closed sets, *Kyungpook Math. J. 33,* 211-219 (1993).
14. M. H. Stone, Applications of the theory of Boolean rings to the general topology, *Trans. A.M.S. 41*, 375-481 (1937).