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Fourier Series Expansions of $\tilde{H}$ –Function

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Abstract

In this Paper we are evaluating sine and cosine series expansions of $\tilde{H}$ –Function defined by Inayet Hussein[4] and employ them to establish certain integral for $\tilde{H}$ –Function.

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1. Introduction

The $\tilde{H}$ –Function a generalization of Fox's H–Function introduced by Inayat Hussain[4] and Studies by Buschman and Shrivastava[1] and other, is defined and represented in the following manner;

$$\tilde{H}_{P,Q}^{M,N}[Z] = \tilde{H}_{P,Q}^{M,N}\left[Z \left| \begin{matrix} (e_j, E_j, \in_j)_{1,N}, (e_j, E_j)_{N+1,P} \\ (f_j, F_j)_{1,M}\ (f_j, F_j, \tau_j)_{M+1,Q} \end{matrix} \right. \right]$$

OR

$$\tilde{H} = {}_{P,Q}^{M,N}\left[Z \left| \begin{matrix} ((\ ))_{1,N}, ((\ ))_{N+1,P} \\ ((\ ))_{1,M}, ((\ ))_{M+1,Q} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_L \overline{\phi}(\xi) z^{\xi} d\xi\ , \qquad (1.1)$$

where,

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{M} \Gamma(f_j - F_j\xi) \prod_{j=1}^{N} \left\{\Gamma(1 - e_j + E_j\xi)\right\}^{\in_j}}{\prod_{j=M+1}^{Q} \left\{\Gamma(1 - f_j + F_j\xi)\right\}^{\tau_j} \prod_{j=N+1}^{P} \Gamma(e_j - E_j\xi)} \qquad (1.2)$$

$J = 1, 2, \ldots\ldots\ldots\ldots, M$ to the right of the path and the singularities of $\left\{\Gamma(1 - e_j + E_j\xi)^{\in_j}\right\}$ $J = 1, 2, \ldots\ldots\ldots\ldots, N$, to the left of the path. The other details about $\tilde{H}$ –Function can be seen in the paper cited earlier, evidently if we take $\in_j (j = 1, \ldots\ldots\ldots, N)$ and $\tau_j (J = M+1, \ldots\ldots\ldots\ldots\ldots, Q)$

equal to unity, The $\tilde{H}$ –Function reduces to well known Fox's H–Function[2].

The following sufficient condition for the absolute convergence of the integral defined in equation (1.1) have been recently given by Gupta, Jain and Agrawal[3]

$$\left.\begin{array}{ll}\text{(i)} & |\arg(z)| < \frac{1}{2}\Omega\pi \quad \text{and} \quad \Omega > 0 \\ \text{(ii)} & |\arg(z)| \le \frac{1}{2}\Omega\pi \quad \text{and} \quad \Omega \ge 0\end{array}\right\}, \tag{1.3}$$

and (a) $\tau \neq 0$ and contour L is so choosen that $(C\tau + \rho + 1) < 0$

(b) $\tau = 0$ and $\rho + 1 < 0$, (1.4)

Where $\Omega = \sum_{j=1}^{M} F_j + \sum_{j=1}^{N} \in_j E_j - \sum_{j=M+1}^{Q} F_j \tau_j - \sum_{j=N+1}^{P} E_j$ (1.5)

$$\tau = \sum_{j=1}^{N} \in_j E_j + \sum_{j=N+1}^{P} E_j - \sum_{j=1}^{M} F_j - \sum_{j=M+1}^{Q} F_j \tau_j \tag{1.6}$$

$$P = \mathrm{Re}\left(\sum_{J=1}^{M} f_j + \sum_{J=M+1}^{Q} F_j \tau_j - \sum_{J=1}^{N} e_j \in_j - \sum_{J=N+1}^{P} e_j\right) + \frac{1}{2}\left(\sum_{J=1}^{N} \in_j - \sum_{J=M+1}^{Q} J_j + P - M - N\right) \tag{1.7}$$

2. The following known result will be utilized in the present paper.

$$\frac{\sqrt{\pi}}{2} \frac{\Gamma(2-s)}{\Gamma\left(\frac{3}{2}-s\right)} (\sin\theta)^{1-2s} = \sum_{r=0}^{\infty} \frac{(s)_r}{(2-s)_r} \sin(2r+1)\theta \tag{2.1}$$

where $0 \le \theta \le \pi$ $\mathrm{Re}(s) \le \frac{1}{2}$ $r = 1, \ldots\ldots, r$

$$\frac{\sqrt{\pi}\,\Gamma(1-s)}{\Gamma\left(\frac{1}{2}-s\right)} \left(\sin\frac{\theta}{2}\right)^{-2s} = 1 + 2\sum_{r=1}^{\infty} \frac{(s)_r}{(1-s)_r} \times \cos r\theta$$

where $0 \le \theta \le \pi$ (2.2)

3. We establish the following fourier series.

Expansions: 1st.

$$\sum_{K=0}^{\infty} \tilde{H}^{M+1,\,N+1}_{P+2,\,Q+2}\left[z \left| \begin{array}{l} (1-r,h:1),\,((\;))_{1,N},\,\ldots\ldots..,((\;))_{N+1,P},(2+r,h) \\ \left(\frac{3}{2},h\right),\,((\;))_{1,M},\,\ldots\ldots..,\,((\;))_{M+1,Q},(1,h,1) \end{array}\right.\right]$$

$$= \frac{1}{2}\sqrt{\pi}\sin\theta\; \tilde{H}^{M,N}_{P,Q}\left[z \left|\sin^{2h}\theta\right| \begin{array}{l} ((\;))_{1,N},\ldots\ldots\ldots..,((\;))_{N+1,P} \\ ((\;))_{1,M},\ldots\ldots\ldots..,((\;))_{M+1,Q} \end{array}\right]$$

Provided h > 0, Re [1 – 2h + 2u $\left(e_j / E_j\right)$] > 0, J = 1, ..., N (3.1)

Expansion: 2nd

$$\tilde{H}^{M+1,\ N}_{P+1,\ Q+1}\left[z \left| \begin{matrix} ((\ ))_{1,N},\ldots\ldots\ldots,((\ ))_{N+1,P},(1,h) \\ \left(\frac{1}{2},h\right)((\ ))_{1,M},\ldots\ldots,((\ ))_{M+1,Q} \end{matrix} \right. \right]$$

$$+2\sum_{r=1}^{\infty}\tilde{H}^{M+1,\ N+1}_{P+2,\ Q+2}\left[z \left| \begin{matrix} (1-r:h:1),((\ ))_{1,N},\ldots\ldots,((\ ))_{N+1,P},(1+r,h) \\ \left(\frac{1}{2},h\right),((\ ))_{1,M},\ldots\ldots,((\ ))_{M+1,Q},(1,h:1) \end{matrix} \right. \right]\times\cos\theta r$$

$$=\sqrt{\pi}\ \tilde{H}^{M,N}_{P,Q}\left[z \left| \left(\sin\frac{\theta}{2}\right)^{2-h} \begin{matrix} ((\ ))_{1,N},\ldots\ldots\ldots\ldots,((\ ))_{N+1,P} \\ \\ ((\ ))_{1,M},\ldots\ldots\ldots\ldots,((\ ))_{M+1+Q} \end{matrix} \right. \right]$$

where h > 0, Re $\left[1-2u+2h\left(e_j/E_j\right)\right]>0,\ j=1,\ldots\ldots,N$

(3.2)

Proof: Expressing the $\tilde{H}$–Function. on the left of (3.1) as Mellin-Barnes type of integral by (1.2) we get,

$$\sum_{r=0}^{\infty}\sin(2r+1)\theta.\frac{1}{2\pi\omega}\int_L \frac{\left(\frac{3}{2}-h\xi\right)\prod_{J=1}^{M}\Gamma\left(f_j-F_j\xi\right)}{\Gamma(h\xi)^1\prod_{J=M+1}^{Q}\Gamma\left(1-f_j+F_j\in_j\right)^{\tau_j}}$$

$$\times\frac{\prod_{J=1}^{N}\Gamma\left(1-e_j+E_j\xi_j\right)^{\in_j}\Gamma(r+h\xi)^1}{\prod_{J=N+1}^{P}\Gamma\left(e_j-E_j\xi_j\right)\Gamma(z+r-h\xi)}z^{\xi}d\xi\times I.$$

Where $I = \dfrac{\Gamma\left(\frac{3}{2} - h\xi\right)\Gamma\left(r + h\xi\right)^{1}}{\Gamma\left(h\xi\right)^{1}\Gamma\left(z + r - h\xi\right)}$

here the path L of integration runs from $C - i\infty$ to $C + i\infty$.

The conditions

$O < C < \dfrac{3}{2h}, \ (h > 0)$

$\operatorname{Re}\left(\dfrac{f_j}{F_j}\right) > C, \ (j = 1, 2, ..., M)$

$\operatorname{Re}\left(\dfrac{e_j - 1}{E_j}\right) < C, \ (j = 1, 2, ..., N)$

ensure that all the poles of $\Gamma\left(\frac{3}{2} - h\xi\right)$ and $\Gamma\left(f_j - F_j\xi\right), (j = 1, 2, ..., M)$ lie to right of L etc.

On changing the order of integration and summation which is easily seen to be justified, the above expression becomes[5,6].

$$\frac{1}{2\pi\omega}\int_L \frac{\Gamma\left(\frac{3}{2} - h\xi\right) \prod_{j=1}^{M}\Gamma\left(f_j - F_j\xi\right) \prod_{j=1}^{N}\Gamma\left(1 - e_j + E_j\xi_j\right)^{\epsilon_j}}{\Gamma\left(2 - h\xi\right) \prod_{j=1}^{Q}\Gamma\left(1 - f_j + F_j\xi\right)^{\tau_j} \prod_{j=N+1}^{P}\Gamma\left(e_j - E_j\xi\right)}$$

$$\times \left\{\sum_{r=0}^{\infty}\frac{\left(h\xi\right)_r}{\left(2 - h\xi\right)_r}\sin\left(2r + 1\right)\theta\right\} z^{\xi} d\xi$$

and using (2.1) we get R.H.S. of (3.1).

(3.1) is the Fourier sine series[5] for the $\tilde{H}$–Function. The Fourier cosine[6] series (3.2) is proved in the analogous manner by using (1.1) and (2.2)

3. Integrals: From (2.1) and (2.2) we easily deduce the integrals

$$\int_0^{\pi} \sin\left(2\pi + 1\right)\theta \sin\theta \quad \tilde{H}_{P,\,Q}^{M,\,N}\left[\frac{Z}{\sin^2 h\theta}\left|\begin{matrix}((\ ))_{1,N}, ...,((\ ))_{N+1,P} \\ ((\ ))_{1,M}, ...,((\ ))_{M+1,Q}\end{matrix}\right.\right] d\theta$$

$$= \sqrt{\pi} \quad \tilde{H}_{P+2,\,Q+2}^{M+1,\,N+1}\left[z \left|\begin{matrix}(1 - r, h), ((\ ))_{1,N}, ..., ((\ ))_{N+1,P}, (2 + r, h:1) \\ \left(\frac{3}{2}, h\right), ((\ ))_{1,M}, ..., ((\ ))_{M+1,Q}, (1, h:1)\end{matrix}\right.\right]$$

$$\text{Where } h>0 \ |\arg\ z|<\frac{1}{2}\Omega\pi,\ r=0,1,\ .... \qquad (3.1)$$

$$\int_0^{\pi}\cos r\theta \quad \tilde{H}_{P,Q}^{M,N}\left[\frac{z}{\sin^2\frac{h\theta}{2}}\left|\begin{matrix}((\ ))_{1,N},\ ....,((\ ))_{N+1,P}\\((\ ))_{1,M},\ ....,((\ ))_{M+1,Q}\end{matrix}\right.\right]d\theta$$

$$=\sqrt{\pi}\quad \tilde{H}_{P+2,\ Q+2}^{M+1,\ N+1}\left[z\left|\begin{matrix}(1-r,h),((\ ))_{1,N},..............,((\ ))_{N+1,P},(2+r,h,1)\\\left(\frac{1}{2},h\right),((\ ))_{1,M},................,((\ ))_{M+1,Q},(1,h,1)\end{matrix}\right.\right]$$

$$\text{provided } h>0 \ |\arg\ z|<\frac{1}{2}\Omega\pi,\ r=0,1,2,.......... \qquad (3.2)$$

Particular cases :

(i) The results (2.1), (2.2) and (3.1) and (3.2) when $\in_j=J_j=1$ (unity) reduce to Fourier series for Fox's H–function.

(ii) The results (2.1), (2.2), (3.1) and (3.2) when $\in_j=\tau_j=1$ and $E_j=F_j=h=1(J=1,........p)\ i=(1,2,.....q)$ reduce to Fourier series for Ω function.

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