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Effect of measurement error on the control chart for mean under known coefficient of variation

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Abstract

In the present paper we investigated the effect of measurement error on the mean ($\bar{X}$) chart with known coefficient of variation (CV). We described the operating characteristic (OC) curve of the $\bar{X}$ chart where both the process average and CV changes.

Key words : Control Charts for Means, Measurement Error, Coefficient of Variation.

1. Introduction

The primary objective of SQC is, the systematic reduction of variability in quality characteristics of the products. To keep the variation under allowable limits, checks are made at two stages. The first stage of check is offered through control chart during production and keeps the manufacturing process stable and capable of operation such that virtually all of the items produced conform to the specification. The research development in the field of SQC has mainly confined to, development of control charts (as a measure of process control) and that of acceptance sampling plans (as a measure of product control). A good deal of literature developed on classical line of Shewhart and Dodge and Romig has already appeared in the text form in the of book by Shewhart[5].

Bayesian approach to control chart problem are based on the the assumption of utilizing a prior information. The parameter $v = \sigma_x/\mu$(coefficient of variation) as a measure of variability of character X, or a measure of dispersion per unit mean, in the lot is one of the important parameter. The problem of estimating v, in fact, assumes a special importance in certain specific situations and is not merely of theoretical interest to avoid of any practical necessity. Searls[4] has provided an estimator $\bar{\bar{x}}$ of μ which utilizes a prior information concerning the value of the population CV. An estimator $\bar{\bar{x}}$ shown to have a smaller mean square (MSE) than the sample mean $\bar{x}$.

Errors of measurements are the differences between observed values recorded under "identical" conditions and some fixed true value. Process measurements which are used in construction of $\bar{\bar{x}}$ Chart involve several sources of error. These include the inherent variability in the process and the error due to the measurement instrument. If the measurement error is large relative to the process variability, the ability of the control chart to detect shifts in process level is affected. The effect of measurement error on the OC of the $\bar{X}$-chart, in cases where only the process average shifts, was discussed by Bennett[2], Abraham[1] and Singh and Soni[6].

Here we have investigated the effect of measurement error on the mean chart with known coefficient of variation. The description is given on OC of the $\bar{\bar{x}}$ chart where both the process average and CV changes. In this development it is assumed that the process has a normal distribution with mean μ and variance σ_p^2. It is further assumed that at the time of determining the control limits the process is in a state of statistical control, and same device is used as will be employed for later measurements.

2. Model Description :

From a random sample of n observations $x_1, x_2, \dots, x_n$ an estimator $\bar{\bar{x}}$ has been constructed by Searls[4], where

$$\bar{\bar{x}} = w \ \Sigma_{j=1}^{n} x_j \ , \tag{2.1}$$

where w is the scalar and is so that the MSE, $E\left(\bar{\bar{x}} - \mu\right)^2$ is minimum. Searls[4] has shown that $w = 1/(n + v^2)$ and hence

$$MSE\left(\bar{\bar{x}}\right) = \sigma^2/(n + v^2). \tag{2.2}$$

Let the relation between the true measurement x and observed measurement X be expressed as $X = x + e$, where e is random error and is independent of *x*. Thus $e \sim N(0, \sigma_e^2)$. Therefore,

$$E(X) = E(x) + E(e) = E(x) = \mu \text{ and}$$
$$V(X) = V(x) + V(e) = \sigma_p^2 + \sigma_e^2 = \sigma_X^2 (say).$$

The relation between size of measurement error r and the correlation coefficient ρ is obtained as

$$\rho = \frac{r}{\sqrt{1+r^2}}, \tag{2.3}$$

where $r = \sigma_p / \sigma_e$. Thus the data used for establishing the limits on the control charts comes from a process that is $N\left(0, \sigma_p^2 + \sigma_e^2\right)$. After some simplification the MSE of observed mean $\bar{\bar{x}}$ is given by

$$MSE(\bar{\bar{x}}) = \mu^2 \left[w^2 n^2 \left(1 + \frac{v^2}{n\rho^2}\right) - 2wn + 1 \right] \tag{2.4}$$

where $v = \sigma_p / \mu$ is the coefficient of variation; The $MSE(\bar{x})$ of equation (2.4) is minimum when $w = \left(1/\left((v^2/\rho^2) + n\right)\right)$. Substituting the value of w in equation (2.4) we get the minimum $MSE(\bar{x})$ as

$$MSE(\bar{x}) = \frac{\sigma_p^2}{(n\rho^2 + v^2)}\,. \tag{2.5}$$

The control chart for mean is set up by drawing the central line at the process average μ and the control limits at $\pm k\,\sigma/\sqrt{n}$, where σ is the process standard deviation and *n* is the sample size. The OC function gives the probability that the control chart indicates the process average as μ when it is actually not μ but $\acute{\mu} = \mu + \gamma\sigma/\sqrt{n + v^2}$ (say) and it is derived by integrating the distribution of mean with $\acute{\mu}$ as the process average between the limits of the control chart. The distribution of sample mean $\bar{x}$ is given by

$$f(\bar{x}) = \frac{\sqrt{n\rho^2 + v^2}}{\sigma_p}\left[\phi\left(\frac{\bar{x} - \mu}{\sigma_p/\sqrt{n\rho^2 + v^2}}\right)\right] \tag{2.6}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}$

Following Montogomery[3] the OC function is obtained, after replacing μ in equation (2.6) by $\acute{\mu}$ and integrating it between the limits of the control chart. After some simplification the OC function is

$$L(p) = \left[\Phi\left\{k\sqrt{\frac{n\rho^2 + v^2}{n}} + \gamma\right\} + \Phi\left\{k\sqrt{\frac{n\rho^2 + v^2}{n}} - \gamma\right\} - 1\right] \tag{2.7}$$

The error of first kind gives the probability of searching for assignable causes when infact, there are no such causes, or in otherwise, it is the probability that the sample values lies out side the control limits when the process average and variation remain unchanged, and it is given by

$$\alpha = 1 - \int_{\mu - k\sigma/\sqrt{n}}^{\mu + k\sigma/\sqrt{n}} f(\bar{x})\ d\bar{x}$$

$$\alpha = 2\Phi\left\{-k\sqrt{\frac{n\rho^2 + v^2}{n}} + \gamma\right\}$$

The 3 σ control limits under measurement error with known CV are

$\bar{x} \pm A_4\sigma_p$,

where $A_4 = 3/\sqrt{n\rho^2 + v^2}$

3. **Numerical illustration and Results**

The values of the error of the first kind are calculated and presented in Table 1 for k=2.0, 2.5, 3.0, n=5, 10, 15, r=2, 4, 6 and ∞ and for different values of v. The different values of OC function for k=3, n=5, r=2, 4, 6 and ∞ and for different values of v are tabulated in Table-2. Table-3 and 4 present the values of control factor A_4 for various values of CV and r=2, 4 and r=6, ∞ respectively.

Table 1. Values of error of first kind under measurement error for k= 2 ,2.5, 3 and different values of r, CV and n

r	v	k=2			k=2.5			k=3		
		n			n			n		
		5	10	15	5	10	15	5	10	15
2	1	0.04551	0.05779	0.06262	0.01242	0.01771	0.01995	0.00270	0.00443	0.00523
	2	0.01141	0.02846	0.03887	0.00157	0.00617	0.00982	0.00015	0.00102	0.00195
	3	0.00126	0.00912	0.01796	0.00006	0.00112	0.00310	0.00000	0.00009	0.00039
	4	0.00006	0.00195	0.00629	0.00000	0.00011	0.00064	0.00000	0.00000	0.00004
	5	0.00000	0.00028	0.00168	0.00000	0.00001	0.00009	0.00000	0.00000	0.00000
4	1	0.03264	0.04128	0.04467	0.00757	0.01075	0.01208	0.00135	0.00221	0.00260
	2	0.00831	0.02055	0.02795	0.00097	0.00379	0.00601	0.00008	0.00051	0.00098
	3	0.00093	0.00665	0.01303	0.00003	0.00069	0.00191	0.00000	0.00005	0.00020
	4	0.00005	0.00143	0.00460	0.00000	0.00007	0.00040	0.00000	0.00000	0.00002
	5	0.00000	0.00021	0.00124	0.00000	0.00000	0.00005	0.00000	0.00000	0.00000
6	1	0.03030	0.03829	0.04142	0.00678	0.00961	0.01080	0.00116	0.00189	0.00222
	2	0.00774	0.01910	0.02596	0.00087	0.00340	0.00538	0.00006	0.00044	0.00084
	3	0.00087	0.00620	0.01213	0.00003	0.00062	0.00172	0.00000	0.00004	0.00017
	4	0.00004	0.00134	0.00429	0.00000	0.00006	0.00036	0.00000	0.00000	0.00002
	5	0.00000	0.00019	0.00116	0.00000	0.00000	0.00005	0.00000	0.00000	0.00000
∞	1	0.02846	0.03594	0.03887	0.00617	0.00874	0.00982	0.00102	0.00165	0.00195
	2	0.00729	0.01796	0.02439	0.00080	0.00310	0.00490	0.00006	0.00039	0.00073
	3	0.00082	0.00584	0.01141	0.00003	0.00057	0.00157	0.00000	0.00004	0.00015
	4	0.00004	0.00126	0.00404	0.00000	0.00006	0.00033	0.00000	0.00000	0.00002
	5	0.00000	0.00018	0.00109	0.00000	0.00000	0.00004	0.00000	0.00000	0.00000

Table 2. Values of OC function under measurement error for n=5, k=3 and different values of CV, ? and r

r	?	v				
		1	2	3	4	5
2	5.0	0.0227	0.1140	0.4354	0.8413	0.9870
	4.5	0.0668	0.2403	0.6321	0.9332	0.9968
	4.0	0.1586	0.4187	0.7988	0.9772	0.9994
	3.5	0.3085	0.6159	0.9094	0.9938	0.9999
	3.0	0.5000	0.7866	0.9669	0.9986	1.0000
	2.5	0.6914	0.9023	0.9903	0.9998	1.0000
	2.0	0.8413	0.9636	0.9977	1.0000	1.0000
	1.0	0.9772	0.9974	0.9999	1.0000	1.0000
	0.0	0.9973	0.9999	1.0000	1.0000	1.0000
4	5.0	0.0363	0.1488	0.4868	0.8654	0.9896
	4.5	0.0976	0.2941	0.6797	0.9457	0.9975
	4.0	0.2132	0.4835	0.8332	0.9824	0.9995
	3.5	0.3839	0.6767	0.9288	0.9954	0.9999
	3.0	0.5811	0.8311	0.9754	0.9990	1.0000
	2.5	0.7595	0.9277	0.9932	0.9998	1.0000
	2.0	0.8858	0.9749	0.9985	1.0000	1.0000
	1.0	0.9862	0.9985	1.0000	1.0000	1.0000
	0.0	0.9986	0.9999	1.0000	1.0000	1.0000
6	5.0	0.0400	0.1574	0.4983	0.8704	0.9901
	4.5	0.1055	0.3066	0.6899	0.9483	0.9977
	4.0	0.2264	0.4978	0.8403	0.9833	0.9996
	3.5	0.4010	0.6896	0.9326	0.9957	0.9999
	3.0	0.5984	0.8400	0.9770	0.9991	1.0000
	2.5	0.7731	0.9325	0.9937	0.9999	1.0000
	2.0	0.8942	0.9770	0.9986	1.0000	1.0000
	1.0	0.9877	0.9986	1.0000	1.0000	1.0000
	0.0	0.9988	0.9999	1.0000	1.0000	1.0000
∞	5.0	0.0433	0.1648	0.5080	0.8746	0.9906
	4.5	0.1124	0.3174	0.6985	0.9503	0.9978
	4.0	0.2377	0.5099	0.8461	0.9841	0.9996
	3.5	0.4154	0.7002	0.9357	0.9960	0.9999
	3.0	0.6127	0.8473	0.9783	0.9992	1.0000
	2.5	0.7842	0.9364	0.9941	0.9999	1.0000
	2.0	0.9008	0.9786	0.9987	1.0000	1.0000
	1.0	0.9889	0.9988	1.0000	1.0000	1.0000
	0.0	0.9990	0.9999	1.0000	1.0000	1.0000

Table 3. Values of Control Factor A_4 for r=2 ,4 and different values of CV and n

n	r=2					r=4				
	v=1	v=2	v=3	v=4	v=5	v=1	v=2	v=3	v=4	v=5
2	1.861	1.268	0.921	0.715	0.582	1.767	1.237	0.909	0.709	0.579
3	1.627	1.186	0.889	0.699	0.573	1.534	1.148	0.872	0.691	0.569
4	1.464	1.118	0.859	0.685	0.565	1.374	1.077	0.840	0.675	0.559
5	1.342	1.061	0.832	0.671	0.557	1.256	1.017	0.810	0.659	0.550
6	1.246	1.011	0.808	0.658	0.550	1.164	0.966	0.784	0.645	0.542
7	1.168	0.968	0.785	0.646	0.542	1.089	0.922	0.760	0.631	0.534
8	1.103	0.930	0.764	0.634	0.535	1.027	0.884	0.738	0.618	0.526
9	1.048	0.896	0.745	0.623	0.529	0.975	0.850	0.718	0.606	0.519
10	1.000	0.866	0.728	0.612	0.522	0.930	0.819	0.699	0.595	0.511
11	0.958	0.839	0.711	0.602	0.516	0.890	0.792	0.682	0.584	0.505
12	0.921	0.814	0.696	0.593	0.510	0.856	0.767	0.666	0.574	0.498
13	0.889	0.791	0.681	0.584	0.504	0.825	0.745	0.651	0.565	0.492
14	0.859	0.770	0.668	0.575	0.499	0.797	0.724	0.637	0.555	0.486
15	0.832	0.750	0.655	0.567	0.493	0.772	0.705	0.624	0.547	0.480
16	0.808	0.732	0.643	0.559	0.488	0.749	0.687	0.612	0.538	0.474
17	0.785	0.715	0.631	0.551	0.483	0.728	0.671	0.600	0.530	0.469
18	0.764	0.699	0.620	0.544	0.478	0.708	0.656	0.589	0.523	0.463
19	0.745	0.685	0.610	0.537	0.473	0.690	0.641	0.579	0.515	0.458
20	0.728	0.671	0.600	0.530	0.469	0.674	0.628	0.569	0.508	0.453
21	0.711	0.658	0.591	0.524	0.464	0.658	0.615	0.559	0.502	0.448
22	0.696	0.646	0.582	0.518	0.460	0.644	0.604	0.550	0.495	0.444
23	0.681	0.634	0.573	0.512	0.455	0.630	0.592	0.542	0.489	0.439
24	0.668	0.623	0.565	0.506	0.451	0.618	0.582	0.534	0.483	0.435
25	0.655	0.612	0.557	0.500	0.447	0.606	0.572	0.526	0.477	0.431

Table 4. Values of Control Factor A_4 for r=6, ∞ and different values of CV and n

n	r=6					r=∞				
	v=1	v=2	v=3	v=4	v=5	v=1	v=2	v=3	v=4	v=5
2	1.748	1.230	0.907	0.708	0.578	1.732	1.225	0.905	0.707	0.577
3	1.515	1.141	0.869	0.690	0.568	1.500	1.134	0.866	0.688	0.567
4	1.356	1.068	0.836	0.673	0.558	1.342	1.061	0.832	0.671	0.557
5	1.239	1.008	0.806	0.657	0.549	1.225	1.000	0.802	0.655	0.548
6	1.147	0.956	0.779	0.642	0.540	1.134	0.949	0.775	0.640	0.539
7	1.073	0.912	0.754	0.628	0.532	1.061	0.905	0.750	0.626	0.530
8	1.012	0.874	0.732	0.615	0.524	1.000	0.866	0.728	0.612	0.522
9	0.960	0.840	0.712	0.603	0.516	0.949	0.832	0.707	0.600	0.514
10	0.916	0.810	0.693	0.591	0.509	0.905	0.802	0.688	0.588	0.507
11	0.877	0.782	0.676	0.581	0.502	0.866	0.775	0.671	0.577	0.500
12	0.843	0.758	0.660	0.570	0.495	0.832	0.750	0.655	0.567	0.493
13	0.812	0.735	0.645	0.560	0.489	0.802	0.728	0.640	0.557	0.487
14	0.785	0.715	0.631	0.551	0.483	0.775	0.707	0.626	0.548	0.480
15	0.760	0.696	0.618	0.542	0.477	0.750	0.688	0.612	0.539	0.474
16	0.737	0.678	0.605	0.534	0.471	0.728	0.671	0.600	0.530	0.469
17	0.716	0.662	0.594	0.526	0.465	0.707	0.655	0.588	0.522	0.463
18	0.697	0.647	0.583	0.518	0.460	0.688	0.640	0.577	0.514	0.457
19	0.680	0.633	0.572	0.511	0.455	0.671	0.626	0.567	0.507	0.452
20	0.663	0.619	0.562	0.504	0.450	0.655	0.612	0.557	0.500	0.447
21	0.648	0.607	0.553	0.497	0.445	0.640	0.600	0.548	0.493	0.442
22	0.634	0.595	0.544	0.491	0.440	0.626	0.588	0.539	0.487	0.438
23	0.620	0.584	0.536	0.484	0.436	0.612	0.577	0.530	0.480	0.433
24	0.608	0.574	0.527	0.478	0.431	0.600	0.567	0.522	0.474	0.429
25	0.596	0.564	0.520	0.472	0.427	0.588	0.557	0.514	0.469	0.424

From Table 2 it is seen that for large values of CV the effect of measurement error on OC is negligible where as the measurement error affects the OC for small values of CV. The measurement depends on the amount of variability in the error terms, and as soon as this amount increases *i.e.* ρ decreases consequently effect of measurement error on the OC function for mean chart with known CV becomes more serious. For instance the value of OC function for error free case , k=3, $\gamma = 0$, v=1 is 0.9999 while keeping other things constant and r=2, 4, 6, the values of OC function are 0.9973, 0.9986 and 0.9988 respectively. It is easily seen that realistic quantities of error have a dramatic effect on the OC curve. From Table-1 the values of error of the first kind for r=∞, 2, 4 and 6 , k=2, n=5 and v=1 are 0.0285, 0.0455, 0.0324 and 0.0300 respectively. The effects of measurement error on the error of the first kind gives clear indication to the producer's, that measurement procedure must be more precise. Table-3 and 4 indicates that when n increases, control factor A_4 decreases. On the other hand when CV increases control factor A_4 decreases, under the measurement error, the control limits tends to become tighter and thereby more sensitive to change in the process. Thus we conclude that control factor is seriously affected under measurement error.

Therefore, we conclude that the effect of measurement error on the OC function of detecting, changes in the process by 3 σ control limits with known coefficient of variation are (i) The $\bar{\bar{x}}$ - chart gives a higher OC function of detecting a change in the process than the traditional $\bar{X}$-chart, (ii) The larger the measurement error, the smaller the detecting OC function. However, this effect is small if n and v is large enough, (iii) If k is small, the measurement error greatly reduces the OC function to detect a change, (iv) Even when the measurement error is large enough to reduce the OC function, the use of larger n and v will result in an improvement, so, the OC function of detecting changes in the existence of the measurement error. The OC depends on the magnitude of the process change; use of a large sample size and CV or a reduction of the measurement error can recover the loss. The selection of either alternative or a combination of both can be made from both economical and technical points of view.

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