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# Finite Sets, Closed Images, and Separation Axioms

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## Abstract

In this paper, additional topological characterizations of finite sets are given using closed images and separation axioms.

*Key words:* finiteness, closed images, and separation axioms.

**Subject classification:** 54A05, 54C10, 54D10, 54D15

## Introduction

1. Within a 2007 introductory graduate level topology class, the students were asked to prove or disprove the open image of a $T_0$ space is $T_0$. The students believed the statement to be false, but unsuccessfully searched for a counterexample. For each unsuccessful search the students used a topological space (X,T) for which X was finite, thus generating the question: "If, in fact, the open image of a $T_0$ space need not be $T_0$, is the open image of a $T_0$ space (X,T), where X is finite, $T_0$?" The investigation of this question has led to many topological characterizations of finite sets using open images and separation axioms[1]. If open images and separation axioms could be used to provide topological characterizations of finite sets, what about closed images and separation axioms? In this paper, this last question is investigated and resolved.

All spaces in this paper are topological spaces.

2. Closed Images and $T_0$ Spaces.

*Definition 2.1.* A function f from a space (X,T) into a space (Y,S) is closed (open) iff for each closed (open) set A in X, f(A) is closed (open) in Y[3].

*Definition 2.2.* A space (X,T) is $T_0$ iff for distinct elements x and y in X, there exits an open set in X containing only one of x and y[3].

In the proof below, the following characterization of $T_0$ spaces will be used.

*Theorem 2.1.* A space (X,T) is $T_0$ iff for distinct elements $x_i$, i = 1, ---, n, where n is a natural number, there exists a closed set C in X containing only one of the n distinct elements[2].

*Theorem 2.2.* Let X be a nonempty set. Then (a) X is finite iff (b) for each $T_0$ topology on X, each closed image of (X,T) is $T_0$.

*Proof:* (a) implies (b): Let T be a $T_0$ topology on X and let f be a closed function from (X,T) onto a space (Y,S). Let u and v be distinct elements of Y and F be all elements in X whose image is u or v. Since X is finite, F is finite. Let C be a closed set in X containing only one of the distinct elements of F. Then f(C) is closed in Y containing only one of u and v and Y\ (f(C)) is open in Y containing only one of u and v.

(b) implies (a): Suppose X is infinite. Let x be in X. Then Z = X\ {x} is infinite and there exists a one-to-one function f from N, the natural numbers, into Z. Let U be the rational numbers in (0,1), which is countable, and let g be a one-to-one function from U onto N. For each r in U, let $x_r = f(g(r))$. For each u in X\ f(g(U)), let $B_u = \{\{u\}\}$. Let $x_r$ be in f(g(U)), let $L_r = \{a : a \text{ is in } U, 0 < a < ½, \text{ and } a < r\}$, let $R_r = \{b: b \text{ is in } U, ½ < b < 1, \text{ and } r < b\}$, and $B_r = \{f(g((a,b))): a \text{ is in } L_r \text{ and } b \text{ is in } R_r\}$. Then the union of $\{B_u$: u is in X\ f(g(U))$\}$ and $\{B_r$: r is in U$\}$ is a base for a $T_0$ topology T on X. Let E = {2n/p: n is in N, p is prime, and 2n/p < 1} and let F = U\ E. Let Y = {c, d, e}, let S = {Ø, Y, {c, d}, {e}}, and let h be the function from (X,T) onto (Y,S) defined by h(v) = e if v is in X\ f(g(U)), h( v) = c if v is in f(g(E)), and h(v) = d if v is in f(g(F)). Since E and F are dense in U[1], then h is closed and open, but (Y,S) is not $T_0$, which is a contradiction. Hence X is finite.

3. Closed Images and $T_1$ Spaces.

In earlier characterizations of finite sets, it was proven that for a nonempty set X the following are equivalent: (a) X is finite, (b) for each $T_1$ topology T on X, each open image of (X,T) is $T_1$, and (c) there is exactly one topology T on X for which (X,T) is $T_1$[1].

As established in the next result, closed images of $T_1$ spaces behave differently from open images of $T_1$ spaces.

*Theorem 3.1.* Let (X,T) be a space. Then (X,T) is $T_1$ iff every closed image of (X,T) is $T_1$.

*Proof:* Suppose (X,T) is $T_1$. Let f be a closed function from (X,T) onto a space (Y,S). Since (X,T) is $T_1$, then singleton sets are closed in X and since f is closed and onto, singleton sets are closed in (Y,S). Thus (Y,S) is $T_1$.

Conversely, suppose every closed image of (X,T) is $T_1$. Since the identity function on (X,T) is closed and onto, then (X,T) is $T_1$.

Even though closed images of $T_1$ spaces behave differently than open images of $T_1$ spaces, closed images of $T_1$ spaces are used below to further characterize finite sets.

*Theorem 3.2.* Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each $T_1$ topology T on X, for each closed image (Y,S) of (X,T), (Y,S) is $T_1$ with S= P(Y), the power set of Y, and (c) for each topology T on X for which (X,T) is $T_1$, each function f from (X,T) onto (Y,S) is closed iff it is open.

*Proof:* (a) implies (b): Let T be a $T_1$ topology on X. Let (Y,S) be a closed image of (X,T). By Theorem 3.1, (Y,S) is $T_1$ and since f is a function from the finite set X onto the set Y, Y is finite. Thus (Y,S) is finite, $T_1$, which implies S = P(Y).

(b) implies (c): Let T be a $T_1$ topology on X. Since the identity function from (X,T) onto itself is closed, then T = P(X). Thus there is exactly one topology on X for which (X,T) is $T_1$, which implies X is finite. Since (X,T) is finite and T = P(X), then the family of closed sets of (X,T), C(T), equals T. Let f be a function from (X,T) onto (Y,S). Since X is finite, then Y is finite. If f is closed, then Y is finite and S = P(Y), which implies C(Y) = S, and for each O in T, O is in C(T), f(O) is closed (Y,S), which is open in (Y,S), and f is open. If f is open, then Y is finite and (Y,S) is $T_1$, which implies C(Y) = S and for each closed set C in X, C is open in X, f(C) is open in Y, which is closed in Y, and f is closed.

(c) implies (a): Suppose X is infinite. Let f be a one-to-one function from N into X and let T be the finite complement topology on X. Then (X,T) is $T_1$. Let Y = {a, b} and let S be the indiscrete topology on Y. Let g be the function from (X,T) onto (Y,S) defined by g(f(2n)) = a for all n in N and for all x not in f(2N), let g(x) = b. Then g is open, but not closed, which is a contradiction. Hence X is finite.

4. *Closed Images and Other Separation Axioms.*

Within Willard's 1970 book[3], an example is given showing the closed continuous image of a Tychonoff space need not be $T_2$. Since Tychonoff implies $T_2$, then, as in the case of $T_0$, the closed image of a $T_2$ space need not be $T_2$. However, as in the case of $T_0$ spaces, $T_2$ and other separation axioms can be used to further characterize finite sets.

In the study of open images, it was proven that a set X is nonempty and finite iff there is exactly one topology T on X for which (X,T) has separation axiom P, where P was replaced by each of $T_1$, $T_2$, Urysohn, $T_3$, $T_{31/2}$, $T_4$, completely normal $T_1$, perfectly normal, and metrizable[1], which will be used below to further characterize nonempty finite sets.

*Theorem 4.1.* Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is metrizable with S= P(Y), (c) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is perfectly normal with S= P(Y), (d) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is completely normal $T_1$ with S= P(Y), (e) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is $T_4$, (f) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is $T_{31/2}$ with S= P(Y), (g) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is $T_3$ with S= P(Y), (h) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is Urysohn, and (i) for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) of (X,T) is $T_2$ with S= P(Y).

*Proof:* (a) implies (b): Let T be a topology on X for which (X,T) is $T_1$ and let (Y,S) be a closed image of (X,T). By Theorem 3.2, S= P(Y), which is a metrizable topology on Y.

Since metrizable implies perfectly normal, which implies completely normal $T_1$, which implies $T_4$, which implies $T_{31/2}$, which implies $T_3$, which implies Urysohn, which implies $T_2$[3], then (b) implies (c), which implies (d), which implies (e), which implies (f), which implies (g), which implies (h), which implies (i).

(i) implies (a): Since $T_2$ implies $T_1$, then for each topology T on X for which (X,T) is $T_1$, each closed image (Y,S) is $T_1$ with S= P(Y) and by Theorem 3.2, X is finite.

Since $T_2$ implies $T_1$, then for each topology T on a nonempty set X for which (X,T) is $T_2$, each closed image (Y,S) is metrizable with S= P(Y). Thus for (a) implies (b) in Theorem 4.1, $T_1$ in part (b) can be replaced by $T_2$. If $T_1$ in Theorem 4.1 is replaced by $T_2$, then, since the identity function form (X,T) onto itself is closed, then (X,T) is $T_2$ with T= P(X), which implies there is exactly one topology on X for which (X,T) is $T_2$ and X is nonempty finite. Thus $T_1$ in the statement of Theorem 4.1 can be replaces by $T_2$.

In a similar manner $T_1$ in the statement of Theorem 4.1 can be replaced by each of the other separation axioms given above the statement of Theorem 4.1 giving many additional topological characterizations of nonempty finite sets.

*Theorem 4.2.* Let X be a nonempty set. Then (a) X is finite iff (b) for each topology T on X for which (X,T) is metrizable, each function f from (X,T) onto a space (Y,S) is closed iff it is open.

*Proof:* (a) implies (b): Let T be a topology on X for which (X,T) is metrizable. Then (X,T) is $T_1$ and by Theorem 3.2 each function f from (X,T) onto a space (Y,S) is closed iff it is open.

(b) implies (a): Suppose X is infinite. Let x be in X, let Z= X\ {x}, let f be a one-to-one function from the natural numbers N into Z, let U be the rational numbers in (0,1), let g be a one-to-one function from U onto N, let $x_r$ = f(g( r)) for each r in U, let E= {2n/p: n is a natural number, p is prime, and 2n/p is in U}, and let F= U\ E as in the proof of Theorem 2.2. Let d be the function from XxX into [0,1] defined by d(a,a)= 0 for all a in X, d(a,b)= d(b,a) for all a and b in X, d(a,b) =1 for all distinct a and b in X with at least one of a and b in X\ f(g(U)), and $d(x_r,x_s)$ equals the absolute value of s-r for all distinct r and s in U. Then d is a metric on X. Let T be the topology on X generated by d, let Y={c,d,e}, let S={Ø, Y,{c,d},{e}}, and let h be the function from X onto Y defined by h(u)= c for all u in f(g(E)), h(u) = d for all u in f(g(F)), and h(u)= e for all u in X\ f(g(U)). Since E and F are dense in U, h is open, but since (Y,S) is not $T_1$, h is not closed. Thus X is finite.

In a similar manner, using the same counterexample, metrizable in Theorem 4.2 can be replaced by each of perfectly normal, completely normal $T_1$, $T_4$, $T_{31/2}$, $T_3$, Urysohn, and $T_2$.

Examples can be easily constructed showing that metrizable in Theorem 4.2 can not be replaced by $T_0$.

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