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Split and non-split dominator chromatic numbers

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Abstract

Graph coloring and graph domination are two major research areas in the field of graph theory. In this paper, we introduce the concept of split and non-split dominator coloring, which combine the concept of domination and coloring and prove some interesting results.

1. Preliminaries

In our study, we consider only simple and undirected graphs. In this section, we review the notions of domination and dominator coloring[1].

Definition 1.1 : Let G = (V, E) be a graph. A subset D of V is called a *dominating set* of G if every vertex in V – D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G.

Definition 1.2 : A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$, is the minimum number of colors required for a proper coloring of G. A *Color class* is the set of vertices, having the same color. The color class corresponding to color i is denoted by V_i.

Definition 1.3: A *dominator coloring* of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. It is implicit that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.

2. Split Dominator Chromatic Number:

In this section, we combine the concept of split domination[2] and dominator coloring, to define split dominator chromatic number $\chi_{sd}(G)$ and obtain $\chi_{sd}(G)$ for some special classes of graphs.

Definition 2.1: Consider a graph G and its dominator coloring with $\chi_d(G)$ colors. The *split dominator chromatic number* of G is the minimum number of color classes to be removed so that the remaining graph of G is disconnected and is denoted by $\chi_{sd}(G)$.

Theorem 2.2: The cycle graph C_n, $n \geq 3$ has $\chi_{sd}(C_n) = \begin{cases} 2 & \text{if } \quad n = 3 \\ 1 & \text{otherwise}. \end{cases}$

Proof : Let C_n be the cycle graph of order $n \geq 3$ and let $v_1, v_2, \ldots, v_n$ be the labels of its vertices. Here n takes one of the three forms 3k, 3k+1 or 3k+2, $k \geq 1$. When k = 1, a dominator coloring for C_3, C_4 and C_5 are obtained by the coloring sequences (1, 2, 3), (1, 2, 1, 2) and (1, 2, 3, 1, 2) respectively. When $k \geq 2$, color the vertices v_{3i-2} by 1, v_{3i-1} by 2 and v_{3i} by (i+2) for i = 1, 2, …, k. When n = 3k+1, the vertex $v_n = v_{3k+1}$ is colored by a new color (k+3) and when n = 3k+2, color the vertices v_{3k+1} and v_{3k+2} by colors 1 and (k+3). For the case n = 3, removal of any two color classes results in a trivial graph and in all other cases, removal of color class 1, results in a disconnected graph. Hence

$$\chi_{sd}(C_n) = \begin{cases} 2 & \text{if } \quad n = 3 \\ 1 & \text{otherwise}. \end{cases}$$

Theorem 2.3 : The path graph P_n of order $n \geq 2$ has $\chi_{sd}(P_n) = 1$.

Proof: Let P_n: $v_1, v_2, \ldots, v_n$ be a path graph of order $n \geq 2$. When $n \leq 8$, a dominator coloring of P_n is obtained by coloring the odd subscripted vertices $v_1, v_3, \ldots$ by color 1 and even subscripted vertices $v_2, v_4, \ldots$ respectively by colors 2, 3, …. When $n \geq 9$ (n = 3k, n = 3k+1 or 3k+2, where $k \geq 3$), color the vertices v_{3i}, i = 1, 2, …, k by 1, v_{3i+1}, i = 0, 1, …, k -1 by 2 and v_{3i+2}, i = 0, 1, …, k – 1 by (i+3). When n = 3k+1, color v_{3k+1} by (k+3), and when n = 3k+2 color v_{3k+1} by 2 and v_{3k+2} by (k+3). It is evident that removal of any one color class results in a disconnected graph. Hence $\chi_{sd}(P_n) = 1$.

Theorem 2.4: The wheel graph $W_{1,\,n}$, n 3 has $\chi_{sd}(W_{1,n}) = \begin{cases} 3 & \text{if } \quad n = 3 \\ 2 & \text{otherwise} \end{cases}$

Proof : Let $W_{1,\,n}$ be a wheel graph with $n \geq 3$. Let the vertices of $W_{1,\,n}$ be labeled as follows. The vertex at the centre is labeled by v_1 and the vertices on the rim are labeled consecutively by $v_2, v_3, \ldots, v_n$. A dominator coloring of $W_{1,\,n}$ is by coloring v_1 by 1 and the vertices in the rim alternatively by 2 and 3 from vertex v_2. When n is odd, the vertex v_n is colored by 4. Now it is seen that the removal of any 3 color classes in the case of n = 3 and color classes 1 and 2 in other cases results in a disconnected graph. Hence

$$\chi_{sd}(W_{1,\,n}) = \begin{cases} 3 & \text{if } \quad n = 3 \\ 2 & \text{otherwise} \end{cases}$$

Theorem 2.5 : The complete graph K_n of order $n \geq 2$ has $\chi_{sd}(K_n) = n-1$.

Proof : A dominator coloring of K_n is by coloring its vertices $v_1, v_2, \ldots, v_n$ by colors 1, 2, 3, …, n respectively. Therefore $\chi_{sd}(K_n) = n - 1$.

Theorem 2.6: The complete bipartite graph $K_{m,\,n}$, m, $n \geq 1$ has $\chi_{sd}(K_{m,\,n}) = 1$.

Proof : In $K_{m,n}$, color the vertices in one of its partition by 1 and the vertices in the other partition by 2. This gives a dominator coloring of $K_{m,n}$. Removal of either of the color class results in a disconnected graph. Hence $\chi_{sd}(K_{m,n}) = 1$.

Note: For any tree T, $\chi_{sd}(T) = 1$.

3. *Non-Split Dominator Chromatic Number:*

In this section, we combine the concept of non-split domination[3], to define non-split dominator chromatic number $\chi_{nd}(G)$ and obtain $\chi_{nd}(G)$ for various classes of graphs.

Definition 3.1: The *non-split dominator chromatic number* of a graph G is the maximum k such that the removal of fewer than k color classes does not split the graph and there is at least one collection of k color classes whose removal results in a connected subgraph of G, and is denoted by $\chi_{nd}(G)$.

Theorem 3.2 : For the cycle graph C_n, $n \geq 3$ and $n \neq 4$, $\chi_{nd}(C_n) = 1$.

Proof: Consider the dominator coloring of C_n as in the proof of theorem 2.2. In this case, removal of color class of the color of vertex v_n results in a connected graph. Hence $\chi_{nd}(C_n) = 1$, when $n \geq 3$ and $n \neq 4$.

Theorem 3.3 : For the wheel graph $W_{1,n}$,

$$n \geq 3,\ \chi_{nd}(W_{1,n}) = \begin{cases} 2 & \text{if } n \text{ is } odd \\ 1 & \text{if } n \text{ is } even \end{cases}$$

Proof : Consider the dominator coloring of $W_{1,n}$ as in the proof of theorem 2.4. It is seen that the removal of color classes 2 and 3, when n is odd and the color class 2 when n is even results in a connected graph. Hence we conclude that $\chi_{nd}(W_{1,n}) = \begin{cases} 2 & \text{if } n \text{ is } odd \\ 1 & \text{if } n \text{ is } even \end{cases}$

Theorem 3.4: For the complete graph K_n of order $n \geq 2$, $\chi_{nd}(K_n) = n-2$.

Proof : A dominator coloring of K_n is by assigning colors 1, 2, 3, …, n respectively to its vertices. Removal of any n-2 color classes results in a connected graph. Therefore $\chi_{nd}(K_n) = n - 2$.

References

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