

Solving multi objective fuzzy fractional programming problem

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Abstract

A method of solving multi objective fuzzy fractional programming problem (MOFFPP) is presented. All the coefficients of the fractional multi objective functions and the constraints are taken to be fuzzy numbers. A numerical example is presented.

Key words : Multi objective fractional programming problem, multi objective fuzzy fractional programming problem, Fuzzy fractional programming, linear ranking function, trapezoidal fuzzy numbers, optimal solution, Crisp LPP.

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1. Introduction

Decision making in case of problems like financial and corporate planning, production planning, marketing and media selection, university planning and student admission, health care and hospital planning, air force maintenance units, bank branches, etc. frequently we may face decision to optimize department/equity ratio, profit / cost, inventory / sales, actual cost / standard cost, out put / employee, student / cost, nurse / patient ratio etc. with respect to some constraints (Lai and Hwang 1996). This type of problem is referred to as fractional programming problem. Most of the

real world problems of above category are characterized by multiple, conflicting and incommensurate aspects of evaluation. These problems are optimized in the framework of multiple objective fractional programming models. Further more while addressing some real world problems, frequently the parameters are imprecise numerical quantities. Fuzzy quantities are very adequate for modeling this situation. Under this situation the above category of problems are termed as multi objective fuzzy fractional programming problems (MOFFPP).

Bellman and Zadeh¹ introduced the concept of fuzzy quantities and also the concept

of fuzzy decision making. A. Munoz, Zimmermann² has introduced fuzzy programming approach to solve crisp multi objective linear programming problem. Mounz⁴, Roubens⁷ reduced fuzzy multi objective linear programmings to crisp problems using ranking function.

In literature, different types of solutions of fractional programming problems have been suggested by many authors such as (Lai and Hwang 1996), (Charnes and Cooper 1962), (Zionts 1968), (Chakrabarty and Gupta 2002), Charnes and Cooper solved linear fractional programming problem by resolving it in to two linear programming problems. Later, Kanti Swarup¹¹ gave an algorithm for the solution of LFPP without reducing it to LPP.

In this paper, we develop a method for solving multi objective fuzzy fractional programming problem where all the parameters of the objective functions and the constraints are fuzzy in nature. In this method, firstly the MOFFPP is reduced to MOFLPP. Similarly the MOFLPP is reduced to MOLPP using the ranking function of Rouben's⁷ then the MOLPP is solved by using fuzzy programming approach of Zimmermann^{2,3}. A numerical example is given at the end to illustrate the method of solution.

2. Linear fractional programming problem:

A linear fractional programming problem is given by

$$\begin{aligned} &\text{Max/min } \frac{c^T x + \alpha}{d^T x + \beta} \\ &\text{Subject to constraints} \\ &A x \leq b, x \geq 0 \end{aligned} \quad (1)$$

Where x , c and d are $n \times 1$ vectors
 A is $m \times n$ matrix
 b is $m \times 1$ vector
 and α, β are scalars.

For maximization problem we have to

maximize the fraction $\frac{c^T x + \alpha}{d^T x + \beta}$. That means

we have to maximize $c^T x + \alpha$ and minimize $d^T x + \beta$. This implies that we have to maximize $(c^T x + \alpha) - (d^T x + \beta)$ which is linear. To sum up the maximization of fractional objective function subject to given constraints of a FPP ultimately reduced to the maximization of a linear objective function subject to same constraints. Similarly for minimization problem the problem reduces to the problem of minimizing $(c^T x + \alpha) - (d^T x + \beta)$.

Definition: X^* is said to be an optimal solution for (1) if $\exists x^* \in X$ such that $F(x^*) \geq F(x)$, for all $x \in X$.

3. Ranking function for fuzzy numbers:

Definition: Let A be a fuzzy number whose membership function can generally be defined as

$$\mu_A(x) = \begin{cases} \mu_A^L(x) & a^1 \leq x \leq a^2 \\ 1 & a^2 \leq x \leq a^3 \\ \mu_A^R(x) & a^3 \leq x \leq a^4 \\ 0 & \text{otherwise} \end{cases}$$

Where $\mu_A^L(x) : [a^1, a^2] \rightarrow [0, 1]$ and $\mu_A^R(x) : [a^3, a^4] \rightarrow [0, 1]$ are strictly monotonic and continuous mappings. Then it is considered as left right fuzzy number. If the membership

function $\mu_A(x)$ is piecewise linear, then it is referred to as a trapezoidal fuzzy number and is usually denoted by $A = (a^1, a^2, a^3, a^4)$.

If $a^2 = a^3$ the trapezoidal fuzzy number is turned into a triangular fuzzy number $A = (a^1, a^3, a^4)$

A fuzzy number $A = (a, b, c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x = b \\ \frac{x-c}{b-c} & b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

Assume that $R: (\mathcal{R}) \rightarrow \mathcal{R}$ be linear ordered function that maps each fuzzy number in to the real number, in which $F(\mathcal{R})$ denotes the whole fuzzy numbers. Accordingly for any two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ we have

$$\tilde{a} \succeq_{\tilde{R}} \tilde{b} \text{ iff } R(\tilde{a}) \geq R(\tilde{b})$$

$$\tilde{a} \succ_{\tilde{R}} \tilde{b} \text{ iff } R(\tilde{a}) > R(\tilde{b})$$

$$\tilde{a} \bar{\succeq}_{\tilde{R}} \tilde{b} \text{ iff } R(\tilde{a}) = R(\tilde{b})$$

We restrict our attention to linear ranking function, that is a ranking function R such that $R(k\tilde{a} + \tilde{b}) = kR(\tilde{a}) + R(\tilde{b})$ for any $\tilde{a}$ and $\tilde{b}$ in $F(\mathcal{R})$ and any $k \in \mathcal{R}$.

Roubens 's ranking function :

The ranking function proposed by F. Roubens is defined by

$$R(\tilde{a}) = \frac{1}{2} \int_0^1 (\inf \tilde{a}_\alpha + \sup \tilde{a}_\alpha) d\alpha.$$

Which reduces to

$$R(\tilde{a}) = \frac{1}{2} (a^1 + a^u + \frac{1}{2}(\beta - \alpha)), \text{ for a trapezoidal}$$

number

$$\tilde{a} = (a^1 - \alpha, a^1, a^u, a^u + \beta)$$

4. Solving fuzzy multi objective linear fractional programming problem :

A fuzzy multi objective linear fractional programming problem is defined as follows⁴⁻⁶

$$\text{Max/ min } \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_p)$$

$$\text{where } \tilde{z}_i = \frac{\tilde{c}_i^T + \tilde{\alpha}_i}{\tilde{d}_i^T + \tilde{\beta}_i}, i = 1, 2, \dots, p \quad (4.1)$$

$$\text{Subject to } \tilde{A}_i x \leq \tilde{b}_i, x \geq 0$$

where $\tilde{c}_i, \tilde{d}_i$ are $(n \times 1)$ fuzzy vectors

$\tilde{A}_i$ are $(m \times n)$ matrices on the fuzzy numbers

$\tilde{b}_i$ are $(m \times 1)$ fuzzy vectors

$\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are fuzzy numbers.

Step-1 :

The fuzzy multi objective linear fractional programming problem (4.1) is first reduced to multi objective fuzzy linear programming problem

$$\text{Max/ min } \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_p)$$

$$\text{Subject to } \tilde{A}_i x \leq \tilde{b}_i, x \geq 0$$

$$\text{where } \tilde{y}_i = (\tilde{c}_i^T + \tilde{\alpha}_i) - (\tilde{d}_i^T + \tilde{\beta}_i)$$

$$\text{It is reduced to the form } \tilde{y}_i = \sum_{j=1}^n \tilde{e}_{ij} x_j + \tilde{\gamma}_i$$

Thus the problem (4.1) is reduced to the multi objective linear programming problem as follows

$$\text{Max/ min } \tilde{Y} = (\sum_{j=1}^n \tilde{e}_{1j} x_j + \tilde{\gamma}_1, \sum_{j=1}^n \tilde{e}_{2j} x_j$$

$$+ \tilde{\gamma}_2, \dots, \sum_{j=1}^n \tilde{e}_{pj} x_j + \tilde{\gamma}_p)$$

$$\text{subject to } \sum_k \tilde{a}_{jk}^i x_k \leq \tilde{b}_j^i, x_k \geq 0$$

(4.2)

In trapezoidal form

$$\tilde{e}_{ij} = (e_{ij}^1, e_{ij}^2, e_{ij}^3, e_{ij}^4)$$

$$\tilde{a}_{jk}^i = (a_{jk}^{i1}, a_{jk}^{i2}, a_{jk}^{i3}, a_{jk}^{i4})$$

$$\tilde{\gamma}_i = (\gamma_i^1, \gamma_i^2, \gamma_i^3, \gamma_i^4)$$

$$\tilde{b}_j^i = (b_j^{i1}, b_j^{i2}, b_j^{i3}, b_j^{i4})$$

Step 2 : Using linear Ranking function R as suggested by Rouben's, the problem (4.2) reduces to

Crisp multiobjective linear programming problem as follows⁸⁻¹⁰

$$\text{Max/min } R(\tilde{Y}) = R(\tilde{y}_1), R(\tilde{y}_2), \dots, R(\tilde{y}_p)$$

$$\text{where } R(\tilde{y}_i) = \sum_{j=1}^n R(\tilde{e}_{ij}) x_j + R(\tilde{\gamma}_i)$$

subject to

$$\sum R(\tilde{a}_{jk}^i) x_k \leq R(\tilde{b}_j^i), \quad x_k \geq 0$$

This again can be written as

$$\text{Max /min } Y = (y_1, y_2, \dots, y_p)$$

$$\text{where } y_i = \sum_{j=1}^n e_{ij} x_j + \gamma_i$$

Subject to

$$\sum a_{jk}^i x_k \leq b_j^i, \quad x_k \geq 0 \quad (4.3)$$

where $e_{ij}, \gamma_i, a_{jk}^i, b_j^i, y_i$, are real numbers

corresponding to the fuzzy numbers $\tilde{e}_{ij}$

$\tilde{\gamma}_i, \tilde{a}_{jk}^i, \tilde{b}_j^i, \tilde{Y}, \tilde{\gamma}_i$ with respect to ranking function, function respectively.

Step 3 : The crisp multiobjective linear programming problem (4. 3) is then solved by using fuzzy programming technique of Zimmermann. The optimal solution thus obtained shall be optimal solution of problem (4.1).

Lemma : The optimal solutions of (4.2) and (4.3) are equivalent.

Proof.

Let M_1, M_2 be sets of all feasible solutions of (4.2) and (4.3) respectively

Then $x \in M_1$ iff $\sum_k (\tilde{a}_{jk}^i) x_k \leq (\tilde{b}_j^i), i=1,2,\dots,m$

By considering R as a linear ranking function, we have

$$\sum_k R(\tilde{a}_{jk}^i) x_k \leq R(\tilde{b}_j^i), \quad i = 1, 2, \dots, m$$

$$\Rightarrow \sum_k a_{jk}^i x_k \leq b_j^i$$

Hence $x \in M_2$

Thus $M_1 = M_2$

Let $x^* \in X$ be the complete optimal solution of (4.2)

Then $\tilde{y}_i(x^*) \geq \tilde{y}_i(x)$, for all $x \in X$, Where X is feasible set of solutions.

$\Rightarrow R(\tilde{y}_i(x^*)) \geq R(\tilde{y}_i(x))$ (applying ranking function R)

$$\Rightarrow \sum_{i=1}^n R(\tilde{e}_{ij}) x_j + R(\tilde{\gamma}_j) \geq \sum_{i=1}^n R(\tilde{e}_{ij}) + R(\tilde{\gamma}_j)$$

$$\Rightarrow \sum_{j=1}^n e_{ij} x_j^* + \gamma_j \geq \sum_{j=1}^n e_{ij} x_j + \gamma_j,$$

$$\forall j = 1, 2, \dots, q$$

$$\Rightarrow y_i(x^*) \geq y_i(x), \quad \forall x$$

5. Numerical Example :

$$\text{Max } \left\{ \frac{6x_1 + 5x_2}{2x_1 + 7}, \frac{2x_1 + 3x_2}{1x_1 + 1x_2 + 7} \right\} \quad (5.1)$$

Subject to

$$1x_1 + 2x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Where

$$\tilde{6} = (5.6, 5.7, 6.2)$$

$$\tilde{5} = (4.7, 4.9, 5.5)$$

(5.2)

$$\begin{aligned}
\tilde{2} &= (1.7, 1.9, 2.5) \\
\tilde{7} &= (6.6, 6.8, 7.2) \\
\tilde{2} &= (1.6, 1.7, 2.2) \\
\tilde{3} &= (2.7, 2.8, 3.1) \\
\tilde{1} &= (0.7, 0.9, 1.5) \\
\tilde{1} &= (0.7, 0.8, 1.1) \\
\tilde{7} &= (6.7, 6.9, 7.5) \\
\tilde{1} &= (0.8, 0.9, 1.4) \\
\tilde{2} &= (1.8, 1.9, 1.4) \\
\tilde{3} &= (2.7, 2.8, 3.1) \\
\tilde{3} &= (2.7, 2.9, 3.5) \\
\tilde{2} &= (1.7, 1.8, 2.1) \\
\tilde{6} &= (5.7, 5.8, 6.1)
\end{aligned}$$

The MOFFPP (5.1) using step -3 reduces to MOFLPP

$$\begin{aligned}
\text{Max } \tilde{y}_1 &= \tilde{6}x_1 + \tilde{5}x_2 - \tilde{2}x_1 - \tilde{7} \\
\text{Max } \tilde{y}_2 &= \tilde{2}x_1 + \tilde{3}x_2 - \tilde{1}x_1 - \tilde{1}x_2 - \tilde{7} \\
\text{Subject to} \\
\tilde{1}x_1 + \tilde{2}x_2 &\leq \tilde{3} \\
\tilde{3}x_1 + \tilde{2}x_2 &\leq \tilde{6} \\
x_1, x_2 &\geq 0
\end{aligned} \quad (5.4)$$

Using ranking function we have

$$y_1 = 5.9x_1 + 5.1x_2 - 2.1x_1 - 6.9 \quad (5.5)$$

$$y_2 = 1.9x_1 + 2.9x_2 - 1.1x_1 - 0.9x_2 - 7.1 \quad (5.6)$$

Subject to

$$1.1x_1 + 2.1x_2 \leq 2.9$$

$$3.1x_1 + 1.9x_2 \leq 5.9, \quad x_1, x_2 \geq 0 \quad (5.7)$$

Solving (5.5) with (5.7) we get

$$x_1 = \frac{344}{221} \quad x_2 = \frac{125}{221}$$

Solving (5.6) with (5.7) we get

$$x_1 = 0 \quad x_2 = \frac{29}{21}$$

The lower bound (L.B) and upper bound (U.B) of objective functions z_1' and z_2' have been computed as follows

Function	U.B	L.B
y_1	8.7995	7.0428
y_2	2.7619	2.3765

Now the problem becomes

$$\text{Min } \lambda = \max - \lambda$$

Subject to

$$3.8x_1 + 5.1x_2 + 1.7567\lambda \geq 8.7995$$

$$0.8x_1 + 2x_2 + 0.3854\lambda \geq 2.7619, \quad x_1, x_2 \geq 0$$

Solving the above problem, the optimal solution is obtained as :

$$x_1^* = 1.1266$$

$$x_2^* = 0.9731$$

Using the above value we get

$$\tilde{x}_1^* = (10.8825, 11.1898, 12.3370)$$

$$\tilde{x}_2^* = (8.5152, 8.9405, 10.0165)$$

$$\tilde{y}_1^* = (4.4299, 4.6398, 5.4951)$$

$$\tilde{y}_2^* = (8.1697, 8.6924, 10.2603)$$

$$\text{Let } \tilde{f}_1^* = \frac{\tilde{x}_1^*}{x_2^*}$$

$$\tilde{f}_2^* = \frac{\tilde{y}_1^*}{\tilde{y}_2^*}$$

The membership functions is defined as follows

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 0 & \text{for } x \leq 10.8825 \text{ and } x > 12.3370 \\ \frac{x - 10.8825}{0.3073} & \text{for } 10.8825 < x \leq 11.1898 \\ \frac{12.3370 - x}{1.1472} & \text{for } 11.1898 < x \leq 12.3370 \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} 0 & \text{for } x \leq 8.5152 \text{ and } x > 10.0165 \\ \frac{x - 8.5152}{0.4253} & \text{for } 8.5152 < x \leq 8.9405 \\ \frac{10.0165 - x}{1.076} & \text{for } 8.9405 < x \leq 10.0165 \end{cases}$$

$$\mu_{\tilde{Y}_1}(x) = \begin{cases} 0 & \text{for } x \leq 4.4299 \text{ and } x > 5.4951 \\ \frac{x - 4.4299}{0.2099} & \text{for } 4.4299 < x \leq 4.6398 \\ \frac{5.4951 - x}{0.8553} & \text{for } 4.6398 < x \leq 5.4951 \end{cases}$$

$$\mu_{\tilde{Y}_2}(x) = \begin{cases} 0 & \text{for } x \leq 8.1698 \text{ and } x > 10.2603 \\ \frac{x - 8.1698}{0.5226} & \text{for } 8.1698 < x \leq 8.6924 \\ \frac{10.2603 - x}{1.5679} & \text{for } 8.6924 < x \leq 10.2603 \end{cases}$$

α - cuts :

$$\alpha \tilde{X}_1 = [0.3073\alpha + 10.8825, 12.3370 - 1.1472\alpha]$$

$$\alpha \tilde{X}_2 = [0.4253\alpha + 8.5152, 10.0165 - 1.076\alpha]$$

$$\alpha \tilde{Y}_1 = [0.2099\alpha + 4.4299, 5.4951 - 0.8553\alpha]$$

$$\alpha \tilde{Y}_2 = [0.5226\alpha + 8.1698, 10.2603 - 1.5679\alpha]$$

$$\alpha(\tilde{X}_1/\tilde{X}_2) = \left[\frac{0.3073\alpha + 10.8825}{10.0165 - 1.076\alpha}, \frac{12.3370 - 1.1472\alpha}{0.4253\alpha + 8.5152} \right] \text{ for } \alpha \in (0, 1]$$

$$\alpha(\tilde{Y}_1/\tilde{Y}_2) = \left[\frac{0.2099\alpha + 4.4299}{10.2603 - 1.5679\alpha}, \frac{5.4951 - 0.8553\alpha}{0.5226\alpha + 8.1698} \right] \text{ for } \alpha \in (0, 1]$$

Membership functions for optimal objective fractions

$$\mu_{\tilde{f}_1^*}(x) = \begin{cases} 0 & \text{if } x \leq 1.0864 \text{ and } x \geq 1.4438 \\ \frac{10.0165x - 10.8825}{0.3073 + 1.076x} & \text{for } 1.0864 < x \leq 1.2516 \\ \frac{12.3370 - 8.5152x}{1.1472 + 0.4253x} & \text{for } 1.2516 < x < 1.4488 \end{cases}$$

$$\mu_{\tilde{f}_2^*}(x) = \begin{cases} 0 & \text{for } x \leq 0.4318 \text{ and } x \geq 0.6726 \\ \frac{10.2603x - 4.4299}{0.2099 + 1.5679x} & \text{for } 0.4318 < x \leq 0.5338 \\ \frac{5.4951 - 8.1698x}{0.8553 + 0.5226x} & \text{for } 0.5338 < x < 0.6726 \end{cases}$$

$$\text{Where } \tilde{f}_1^* = \frac{\tilde{X}_1^*}{\tilde{X}_2^*}, \quad \tilde{f}_2^* = \frac{\tilde{Y}_1^*}{\tilde{Y}_2^*}$$

Conclusion

The method adopted here for reducing the multi objective fuzzy linear fractional programming problem to that of multi objective fuzzy linear programming problem and then reducing it to crisp one using ranking function is a new approach to tackle a MOFLFPP. This approach can be extended to multi objective fuzzy non linear fractional programming problem.

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