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Roman lict domination in graphs

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Abstract

The lict graph $n(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident.

A Roman dominating function on a lict graph $n(G) = (V', E')$ is a function $f : V' \to \{0,1,2\}$ satisfying the condition that every vertex u' for which $f(u')=0$ is adjacent to at least one vertex v' for which $f(v')=2$. The weight of a Roman dominating function is the value $f(V') = \sum_{u' \in V'} f(u')$. The minimum weight of a Roman dominating function on a lict graph $n(G)$ is called the Roman lict domination number of G and is denoted by $\gamma_{Rn}(G)$.

In this paper we study the graph theoretic properties of $\gamma_{Rn}(G)$ and its exact values for some standard graphs and expressed in terms of members of G but not the members of $n(G)$. Also we establish the some relations with other domination parameters.

Key words: Lict graph/ Dominating set/ Roman dominating set/ Roman domination number.

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Introduction

In this paper, we follow the notation and terminologies of Harary[2]. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. A vertex is called a cut vertex if removing it from G increases the number of components of G. A set $S \subseteq V$ is a dominating set if each vertex in V is dominated by at least one vertex of S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set.

In general we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices of X. $N(v)$ And $N[v]$ denotes the open and closed neighborhoods of a vertex v. The notation $\alpha_0(G)(\alpha_1(G))$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. Also $\beta_0(G)(\beta_1(G))$ is the maximal independent set of vertex (edge) of G.

A Spider is a tree with the property that the removal of all end paths of length two of T results in an isolated vertex, called the head of a Spider. Similarly an Octopus is a tree with the property that the removal of all end paths of length three of T results in an isolated vertex, called the head of an Octopus. Obviously the tentacle of an Octopus is an end path of length three and the tentacle of a Spider is an end path of length two[1-5].

A caterpillar is a tree in which removal of all end vertices of T results in a path.

The lict graph $n(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident[6].

The definition of a Roman dominating function is given implicitly in [1,4]and[8]. For a graph $G=(V,E)$, let $f : V \rightarrow \{0,1,2\}$, and let (F_0,F_1,F_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \setminus f(v) = i\}$ and $|V_i| = n_i$, for i=0,1,2. Note that there exist 1-1 correspondence between the functions $f : V' \rightarrow \{0,1,2\}$ and the ordered partitions (F_0,F_1,F_2). Thus, we will write $f=(F_0,F_1,F_2)$. A Roman dominating function (RDF) on a graph $G=(V,E)$ is a function $f : V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of a RDF is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of RDF on a graph G is called the Roman domination number and is denoted by $\gamma_R(G)$.

We now define the Roman lict domination number of a graph G as follows[7].

For a lict graph $n(G)=(V',E')$, let $f : V' \rightarrow \{0,1,2\}$, and let (F'_0, F'_1, F'_2) be the ordered partition of V' induced by f, where $V'_i = \{v' \in V' \setminus f(v') = i\}$ and $|V'_i| = n'_i$, for i=0,1,2. Note that there exist 1-1 correspondence between the functions $f : V' \rightarrow \{0,1,2\}$ and the ordered partitions (F'_0, F'_1, F'_2). Thus, we will write $f = (F'_0, F'_1, F'_2)$. A Roman dominating function (RDF) on a lict graph $n(G)=(V',E')$ is a function $f : V' \rightarrow \{0,1,2\}$

satisfying the condition that every vertex u' for which $f(u')=0$ is adjacent to at least one vertex v' for which $f(v')=2$. The weight of a Roman domination function is the value $f(V')=\sum_{u'\in V'} f(u')$. The minimum weight of a Roman dominating function on a lict graph $n(G)$ is called the Roman lict domination number of G and is denoted by $\gamma_{Rn}(G)$.

Results

We need the following results for our further results.

Theorem A [3]: For any (p,q) graph G, $\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \gamma(G)$.

Theorem B [5]: For any (p,q) graph G, $p-q \leq \gamma(G)$.

Theorem C [5]: If G is a connected graph, then $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \gamma(G)$.

Theorem D [5]: For any graph G with end vertex, $\gamma(G)=\gamma_s(G)$.

Theorem E [7]: For any connected (p,q) graph G, $\gamma(G^2) \leq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil$.

Theorem F [7]: For any graph G, $\gamma(G^2)+\gamma(G) \leq \gamma_t(G)+1$.

In the following theorem, we establish the exact values of $\gamma_{Rn}(G)$ for some standard graphs.

Theorem [1]:

(1) For any path P_p, with $p\geq 3$ vertices, $\gamma_{Rn}(P_p)$

$= p-1$, if $p = 3,5$

$=p-2$, if $p=4$ and $p\geq 6$.

(2) For any cycle C_p, with $p \geq 3$ vertices, $\gamma_{Rn}(C_p)=\frac{2p}{3}$, if $p \equiv 0 (\mathrm{mod}\, 3)$.

$=\left\lceil \frac{2p}{3} \right\rceil$, otherwise.

(3) For any wheel W_p, with $p \geq 4$ vertices, $\gamma_{Rn}(W_p)=\left\lceil \frac{2p}{3} \right\rceil$.

(4) For any complete graph K_p, $\gamma_{Rn}(K_p)=p-1$.

(5) For any star $K_{1,n}$, $\gamma_{Rn}(K_{1,n})=2$.

The Roman domination number of a Spider H as $\gamma_{Rn}(H)$ has both the lower bound and upper bound with respect to the $\gamma_R(H)$. Now in the following theorem we put the bound on tentacles of a Spider and to obtain the lower bound for $\gamma_{Rn}(H)$.

Theorem [2]: If $G=H$ is any Spider with at least two healthy tentacles then $\gamma_{Rn}(H)\geq\gamma_R(H)$. Equality holds if H is a Spider with exactly two healthy tentacles.

Proof: Let $G=H$ is a Spider with at least two healthy tentacles. Assume H is a healthy or wounded Spider. Then $f=(F_0, F_1, F_2)$ be a Roman dominating function with Roman dominating set D in H. Let $f'=(F'_0, F'_1, F'_2)$ be the corresponding Roman dominating function with Roman dominating set D in $n(H)$. Suppose v be the head of the Spider and $A=\{v_1, v_2,........,v_n\}$ be the set of end vertices of healthy tentacles in H. Then $\{v\}\in F_2$, $\{A\}\in F_1$ and $N(v)\in F_0$. Hence $D=F_0\cup F_1\cup F_2$. In, $n(H)$, suppose $E=\{e_1, e_2,........,e_n\}$ be the set of non end edges in H then $\{E\} \subseteq V[n(H)]$ such that $\forall e_{i,} where 1 \le i \le n$ is a cut vertex of $n(H)$ with maximum degree and $\{E\} \in F'_2$ and $V[n(H)]-\{E\} \in F'_0$. Clearly $D' = F'_0 \cup F'_2$, which gives the required result.

Again we have the upper bound for $\gamma_{Rn}(H)$. To establish the upper bound for $\gamma_{Rn}(H)$. we apply the bound on the tentacles of a Spider and obtain the result in the following theorem.

Theorem [3]: If H is a Spider with n-tentacles such that n-1 tentacles are wounded then $\gamma_{Rn}(H) < \gamma_R(H)$.

Proof: Let $f=(F_0,F_1,F_2)$ be a Roman dominating function in H and $f=(F'_0, F'_1, F'_2)$ be a corresponding Roman dominating function in $n(H)$. Suppose statement of the theorem holds, then $\{v\} \in F_2$ where v is the head of the Spider, $N(v) \in F_0$ and $\{v_1\} \in F_1$ where v_1 is the end vertex of the healthy tentacle such that $\gamma_R(H) = F_0 \cup F_1 \cup F_2 = 3$ is a constant positive integer for any value of n. Since there exist exactly one non end edge $\{e_1\}$ in $n(H)$. Clearly $\{e_1\} \in F'_2$ and $V[n(H)]-\{e_1\} \in F'_0$. Hence $\gamma_{Rn}(H) = F'_0 \cup F'_2 = 2$. Which gives $\gamma_{Rn}(H) < \gamma_R(H)$.

In view of Theorem[2] and Theorem[3], we have the following result.

Corollary: If H is a Spider with all wounded tentacles, then $\gamma_{Rn}(H) = \gamma_R(H)$.

In the following theorem we develop the result on a graph $G=T$ which is an octopus.

Theorem [4]: For any octopus T with n-tentacles, $\gamma_R(T) = \gamma_{Rn}(T)$. If T is

(1) a healthy octopus.

Or

(2) a wounded octopus with exactly one wounded tentacle of length two and does not contains a wounded tentacle of length one.

Or

(3) a wounded octopus with exactly two wounded tentacles of length two and at most n wounded tentacles of length one.

Or

(4) a wounded octopus with only wounded tentacles of length one.

Proof:

Let $f=(F_0,F_1,F_2)$ be a Roman dominating function with Roman dominating set as D in an octopus T and $f = (F'_0, F'_1, F'_2)$ be a corresponding Roman dominating function with Roman dominating set as D' in $n(T)$. Let v be a head of the octopus.

For (1): Suppose statement of the theorem holds then $\{v\} \in F_1$, $N(A) \in F_2$ where A is the set of all end vertices in T and $\{A \cup N(v)\} \in F_0$ such that $D = F_0 \cup F_1 \cup F_2$.

In $n(T)$, the set of all non end edges $\{B_i; 1 \le i \le n\}$ incident with $N(A)$ in T belongs to $F'_2, v \in F'_1$, and $\{V[n(T)]-(v \cup \{B_i\}\}$ $\in F'_0$. Clearly $D' = F'_0 \cup F'_1 \cup F'_2$. Hence, $D=D'$ which gives $\gamma_R(T) = \gamma_{Rn}(T)$.

For the conditions 2 and 4, we use the contradiction.

For (2): Suppose $\gamma_R(T) = \gamma_{Rn}(T)$ and T has exactly one wounded tentacle of length two and at least one wounded tentacle of length one. Then in T, $v \cup N(S) \in F_2$, Where S is the set of all end vertices of the healthy tentacle, $N(v) \cup S \in F_0$ and $u \in F_1$, Where is the end vertex of the wounded tentacle of length two. Hence $D = F_0 \cup F_1 \cup F_2$. In $n(T)$ the set of all non end edges $\{E_i; 1 \le i \le n\}$ incident with $\{N(S) \cup N(u)\}$ belongs to F'_2 and $\{V[n(T)]\} - \{E_i\}\} \in F'_0$. Hence $D' = F'_0 \cup F'_2$, which gives $\gamma_R(T) > \gamma_{Rn}(T)$, a contradiction[1-8].

For (3): Suppose U and W be the set of end vertices of tentacles of length three and two respectively in T. Then $\{N(U) \cup v\} \in F_2$, $\{W\} \in F_1$ and $\{U \cup N(v)\} \in F_0$. Hence $D = F_0 \cup F_1 \cup F_2$. In n(T), the set of non end edges $\{E_k; 1 \le i \le n\}$ incident with $\{N(U) \cup N(W)\}$ belongs to F'_2 and $\{V[n(T)]\} - \{E_k\}\} \in F'_0$. Hence $D' = F'_0 \cup F'_2$, which gives $\gamma_R(T) = \gamma_{Rn}(T)$.

For (4): Let $H_1 = \{t_1, t_2, ..., t_i\}$ where $1 \le i \le n$ be the set of tentacles of length one. $H_2 = \{t_1, t_2, ..., t_j\}$ Where $1 \le j \le n$ be the set of tentacles of length two and $H_3 = \{t_1, t_2, ..., t_k\}$ where $1 \le k \le n$ be the set of healthy tentacles such that $H_1 \cup H_2 \cup H_3 = E(T)$.

We consider the following cases.

Case (1): Suppose $H_1 = \phi$ and $H_2 \ne \phi$ such that there exist at least one element $t_i \in H_2$ and if i=1, then the condition (2) holds.

Case (2): Suppose $H_1 = \phi$ or $H_1 \ne \phi$ and $H_2 \ne \phi$ such that there exist at least one element $t_j \in H_2$ and j = 1,2 then the result gives the condition (3).

Case (3): Suppose $H_1 = \phi$ or $H_1 \ne \phi$ and $H_2 \ne \phi$ such that there exist at least one element $t_k \in H_2$ such that k = 3,4,.... . Then $\{v \cup N(U)\} \in F_2, \{W\} \in F_1$ and $\{N(v) \cup U\} \in F_0$ where U and W be the set of and vertices of tentacles of length three and two respectively such that $D = F_0 \cup F_1 \cup F_2$.

In $n(T)$, the set of non end edges $\{E_f; 1 \le f \le n\}$ incident with $\{N(U) \cup N(W)\}$ $\in F_0$ which belongs to F'_2 and $\{V[n(T)] - \{E_f\}\} \in F'_0$ such that $D' = F'_0 \cup F'_2$ which gives $\gamma_R(T) < \gamma_{Rn}(T)$ a contradiction.

For converse we give the proof for condition (2) and the condition (4).

For the condition (2),

Assume S_1 be the set of all end vertices of the tentacles then $N(S_1) \in F_2$ and $\{V[T] - N(S_1)\} \in F_0$ such that $D = F_0 \cup F_2$. In $n(T)$, the set of non end edges (E_j) where $1 \le j \le n$, incident with $N(S_1)$ belongs to F'_2 and

$\{V[n(T)]-\{E_j\}\}\in F'_0$ such that $D'=F'_0\cup F'_2$, which gives $\gamma_R(T)=\gamma_{Rn}(T)$.

For the condition (4): Let S_2 be the set of all end vertices of healthy tentacles then $\{v\cup N(S_2)\}\in F_2$ and $\{N(v)\cup\{S_2\}\}\in F_0$ such that $D=F_0\cup F_2$. In $n(T)$, suppose $\{H_i;1\le i\le n\}$ be the set of non end edges incident with $N(S_2)$ then $\{v\cup H_i\}\in F'_2$ and $\{V[n(T)]-\{v\cup\{H\}_i\}\}\in F'_0$ such that $D'=F'_0\cup F'_2$, which gives $\gamma_R(T)=\gamma_{Rn}(T)$.

Theorem [5]: For any octopus T with n-tentacles $\gamma_R(T)>\gamma_{Rn}(T)$. If T is a wounded octopus with exactly one wounded tentacle of length two and at least one wounded tentacle of length one[8].

Proof: Let $f:V\to\{0,1,2\}$; be a RDF with $D=(F_1\cup F_2)$ as a RDS of T and $f:V'\to\{0,1,2\}$; be RDF with $D'=(F'_1\cup F'_2)$ as a RDS of $n(T)$. Also both D and D' the set F_0 and F'_0 respectively in T and $n(T)$.

Suppose v be a vertex with maximum degree in T, then $v\in F_2$. If and v_i where $1\le i\le n$ be the set of end vertices of the tentacles of length two three respectively. Then $u\in F_1$, $N(v_i)\in F_2$ and $\{N(v)\cup v_i\}\in F_0$. Since there is exactly one wounded tentacle of length two and at least one wounded tentacles of length one, there exist $e_1=u_1$ and $e_j=u_j$ be the set of non end edges incident with $N(u)$ and $N(v_i)$ respectively such that $\{u_1\cup u_j\}\in F'_2$; $1\le i\le n$. Hence $\gamma_R(T)>\gamma_{Rn}(T)$.

Theorem [6]: For any octopus T with n-tentacles $\gamma_{Rn}(T)>\gamma_R(T)$. If T is a wounded octopus with at least three wounded tentacles of length two[5-8].

Proof: Suppose T be a wounded octopus with $\gamma_{Rn}(T)>\gamma_R(T)$ and has one or two wounded tentacles of length two. Then by the Theorem (4), we obtain the equality result, which is a contradiction to our supposition. Now assume that T is a wounded octopus with at least three wounded tentacles of length two. Let v_i and u_j where $3\le i\le n$ and $1\le j\le n$ be the set of end vertices of tentacles of length two and three respectively. Then $v_i\in F_1$, $\{v\cup N(u_j)\}\in F_2$ and $\{N(v)\cup u_j\}\in F_0$ where $v\in\Delta(G)$. Clearly $\gamma_R(T)=|F_0\cup F_1\cup F_2|$.

Now $\{e_i=w_{i,}e_j=w_j\}\in F'_2$ be the set of non end edges incident with $N(v_i)$ and $N(u_j)$ respectively such that $\{e_i\cup e_j\}\in V[n(T)]$ and $V[n(T)]-\{e_i\cup e_j\}\in F'_0$. Hence $\gamma_{Rn}(T)=|F'_0\cup F'_2|$.

If the number of wounded tentacles of length two increases in T then $\gamma_R(T)$ is an increase by 1 for each tentacle where as the $\gamma_{Rn}(T)$ increases by 2 for corresponding wounded tentacles of length two in $n(T)$.

Theorem [7]: For any (p,q) graph G with $p\ge 3$ vertices $\gamma_{Rn}(G)\ge\left\lceil\frac{p}{\Delta(G)}\right\rceil$.

Equality holds if $G \cong K_{1,n}, P_4$ and C_3.

Proof: Let $f : V' \to \{0,1,2\}$ be a RDF with a RDS D' in $n(G)$ such that $D' = (F'_0 \cup F'_1 \cup F'_2)$ and $V[n(G)] = |E \cup C|$ where E and C are the set of edges and cut vertices of G.

Suppose $C_1 = \{v_1, v_2, \ldots\ldots\ldots v_n\}$ be the set of all non end vertices in G then there exist at least one vertex of maximum degree $\Delta(G)$ in C_1 such that $|D|.\Delta(G) \geq P$. It follows that

$$\gamma_{Rn}(G) \geq \left\lceil \frac{p}{\Delta(G)} \right\rceil.$$

For the equality we consider the following cases.

Case (1): Suppose G is isomorphic to a star, then $n(G) \cong K_p$. Clearly $\gamma_{Rn}(G) = 2$. Since for any star $K_{1,n}$, $p = \Delta(G) + 1$ and $\left\lceil \frac{p}{\Delta(G)} \right\rceil = 2$. Hence it follows that

$$\gamma_{Rn}(G) = \left\lceil \frac{p}{\Delta(G)} \right\rceil.$$

Case (2): Suppose $G = P_4$ or C_3. Then from Theorem (1), $\gamma_{Rn}(G) = 2$ and we have $\left\lceil \frac{p}{\Delta(G)} \right\rceil = 2$. Hence $\gamma_{Rn}(G) = \left\lceil \frac{p}{\Delta(G)} \right\rceil$.

Theorem [8]: For any graph G with $p \geq 3$ vertices $\gamma'(G) + 1 \leq \gamma_{Rn}(G)$ where $\gamma'(G)$ is the edge domination number of G.

Proof: For a graph $G = (V, E)$. Let $E' = \{e_1, e_2, \ldots\ldots\ldots e_n\}$ be an edge dominating set of G and $C = \{c_1, c_2, \ldots\ldots\ldots, c_n\}$ be the set of all cut vertices in G. Let $f : V' \to \{0,1,2\}$ be a RDF in $n(G)$ and $E(G) \cup C(G) = V[n(G)]$. Clearly $|E' \cup C| \subseteq V[n(G)]$. Also E' belongs to F'_2 or F'_1 in $n(G)$ such that $|F'_1 \cup F'_2|$ is a roman lict domination number of G. Hence $\gamma'(G) + 1 \leq \gamma_{Rn}(G)$.

Theorem [9]: For any (p,q) graph G, $\gamma_{Rn}(G) \geq \alpha_0(G)$.

Proof: Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ where $1 \leq i \leq n$ be the vertex cover of G such that $|S| = \alpha_0(G)$. Now let $D' = \{v_1, v_2, \ldots\ldots\ldots, v_k\}$ be the minimal roman dominating set of $n(G)$. Since $E(G) \cup C(G) = V[n(G)]$. It follows that $D' \supseteq S$. Clearly $|D'| \geq \alpha_0(G)$. Hence $\gamma_{Rn}(G) \geq \alpha_0(G)$.

Theorem [10]: If T is a tree with every non end vertex is adjacent to at least one end vertex, then $\gamma_{Rn}(T) \leq P - C + 1$.

Proof: Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be the set of all cut vertices in T such that $|S| = C$. Now without loss of generality, let $f : V_i \to \{0,1,2\}; i = 0,1,2$ and (F'_0, F'_1, F'_2) be the ordered partitions of V' induced by f with $|V_i'| = n_i$ for i=0,1,2. Since the set F'_2 dominates F'_0, the set $D' = (F'_1 \cup F'_2)$ is a roman dominating set of $n(T)$. Further each block in $n(T)$ is a complete block and the number of cut vertices in T is equal to the number of blocks in $n(T)$. Clearly it follows that $|D'| \le P - |S| + 1$. Therefore $\gamma_{Rn}(T) \le P - C + 1$.

Theorem [11]: For any connected (p,q) graph G with $G \ne P_5$, $\gamma_{Rn}(G) \le q - \Delta'(G) + 1$ where $\Delta'(G)$ is the maximum edge degree in G.

Proof: Let $D' = (F'_1 \cup F'_2)$ be a RDS with minimum number of vertices in $n(G)$ having one vertex in D' and other in $F'_0 = V' - D'$. Suppose 'e' be an edge of maximum degree $\Delta'(G)$ in G. Then $e = v' \in D'$ in $n(G)$. Further if for every vertex $u' \in N(v')$ is adjacent to a vertex w' which is not adjacent to v' in $n(G)$. Then it follows that $V[n(G)] - \{N(v') \cup (w')\}$ is a roman dominating set of $n(G)$. Hence $\gamma_{Rn}(G) \le q - \Delta'(G) + 1$. Suppose, $G = P_5$, let $\{e_1, e_2, e_3, e_4\}$ be the set of edges and $\{c_1, c_2, c_3\}$be the set of cut vertices of P_5. Then $\{(e_1, e_2, e_3, e_4) \cup (c_1, c_2, c_3)\} \in V[n(G)]$. Now $\gamma_{Rn}(P_5) = 4$. But $q - \Delta'(G) + 1 = 3$ which gives the contradiction result to the statement. Hence $G \ne P_5$. This completes the proof.

Theorem [12]: For any graph G with $p \ge 3$, $\gamma_t(G) \le \gamma_{Rn}(G)$.

Proof: Suppose D is a dominating set of G, then $D \cup H$ is a total dominating set of G, where $H \in N(D)$ and $H \subseteq V - D$. Let $f : V' \to \{0,1,2\}$ be a RDF with RDS D' in $n(G)$. Let $F = \{e_1, e_2, e_3, \ldots\ldots, e_n\}$ be the set of edge in G, which is an dominating set of G and $S = \{c_1, c_2, \ldots\ldots, c_n\}$ be the set of cut vertices of G such that $S \subseteq D \cup H$. Since $S \cup F \subseteq V[n(G)]$. The edges incident with $\{D \cup H\}$ together with F forms RDS D' in $n(G)$. Further the edges $\{e_i\}$ where $1 \le i \le n$ which are incident with S belongs to F'_2 and the non end edges which are not adjacent to $\{e_i\}$ belongs to F'_1 and remaining vertices of $n(G)$ are dominated by F'_2 such that $D' = (F'_1 \cup F'_2)$. Clearly $|D \cup H| \le |D'|$. Hence $\gamma_t(G) \le \gamma_{Rn}(G)$.

Theorem [13]: For any (p,q) graph G, $\gamma(G) \le \gamma_{Rn}(G)$. Equality holds if and only if G is K_2.

Proof: Let $f = (F'_0, F'_1, F'_2)$ be a Lict roman dominating function. Since F'_2 dominates F'_0, $(F'_1 \cup F'_2)$ is a roman dominating set of $n(G)$. Hence $\gamma(G) = (F'_1 \cup F'_2)$

$$= |F'_1| \cup |F'_2|$$
$$\le |F'_1| + 2|F'_2|$$
$$= \gamma_{Rn}(G)$$

For equality, suppose $G \ne K_2$. Then consider the following cases.

Case (1): Assume G is $K_{1,2}$. Then from Theorem [1], $\gamma_{Rn}(G)=2$. But $\gamma(G)=1$, a contradiction.

Case (2): Assume G is K_3. Then again from Theorem[1], $\gamma_{Rn}(G)=2$. and $\gamma(G)=1$, which gives a contradiction.

For the converse, suppose $G=K_2$, then one can easily verify that $\gamma(G)=\gamma_{Rn}(G)$.

Theorem [14]: For any graph G, $\gamma(G^2)+\gamma(G)-1\leq\gamma_{Rn}(G)$.

Proof: The proof of this Theorem follows from The Theorem [12] and Theorem [F]; $\gamma(G^2)+\gamma(G)\leq\gamma_t(G)+1$.

Theorem [15]: For any (p,q) graph G, $p-q\leq\gamma_{Rn}(G)$.

Proof: The proof of the theorem follows from Theorem [13] and Theorem [B]; $p-q\leq\gamma(G)$.

Theorem [16]: If G is a connected graph, then $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq\gamma_{Rn}(G)$.

Proof: It follows from Theorem [13] and Theorem[C]; $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq\gamma(G)$.

Theorem [17]: For any (p,q) graph G, $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq\gamma_{Rn}(G)$.

Proof: It follows from the Theorem [13] and Theorem [A]; $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq\gamma(G)$.

Theorem [18]: For any graph G, $\gamma_s(G)\leq\gamma_{Rn}(G)$.

Proof: From Theorem [13], we have
$\gamma(G)\leq\gamma_{Rn}(G)$ .... (1)
Also from Theorem [D], $\gamma(G)=\gamma_s(G)$ (2)
From equation (1) and (2), we get
$\gamma_s(G)\leq\gamma_{Rn}(G)$.

Theorem [19]: For any connected (p,q) graph G, $\gamma(G^2)\leq\gamma_{Rn}(G)$.

Proof: Since from Theorem [17], we have $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq\gamma_{Rn}(G)$ .... (1)
Also from Theorem [E], we have
$\gamma(G^2)\leq\left\lceil \frac{p}{\Delta(G)+1} \right\rceil$.............(2)
So from equation (1) and (2). We obtain $\gamma(G^2)\leq\gamma_{Rn}(G)$.

Theorem [20]: For any non trivial tree T $\gamma_{Rn}(T)\leq\beta_0(T)+\gamma(T)-1$.

Proof: $f: V \to \{0,1,2\}$ be a RDF with RDS D' in $n(T)$. Let $E = (e_1, e_2, ..., e_n)$ and $C = (c_1, c_2, ..., c_n)$ be the set of edges and cut vertices of T respectively then $|E| \cup |C| = V[n(G)]$.

Let D be a minimal dominating set of T and B be a maximum independent set of T. Since B is a dominating set of T such that $|D| \le |B|$ and $|D| \le \gamma_{Rn}(T)$. Clearly $\gamma_{Rn}(T) \le |D \cup B - 1|$. Hence $\gamma_{Rn}(T) \le \beta_0(T) + \gamma(T) - 1$.

Lemma: If T is any caterpillar with at least three n codes such that $(c_2, c_3, ..., c_{n-1})$ codes are zero, then $\gamma_{Rn}(T) \ge \gamma_R(T)$.

Proof: Let $f: V \to \{0,1,2\}$ and $f: V' \to \{0,1,2\}$ be a roman dominating functions with roman dominating sets D and D' respectively. Suppose T is a caterpillar with $\{C_n\}$ codes where $n \ge 3$ and $(c_2, c_3, ..., c_{n-1}) = 0$.

Then we have following cases.

Case (1): If the codes $C_n = 3k$; $k = 1,2,....$, then $\{C_n, C_{n-1}\} \in F_2$
where $i = 3h - 1$; $h = 1,2,.....$

Case (2): If $C_n = 3k+1$; $k = 1,2,....$, then $\{C_n, C_{n-1}\} \in F_2$ where $j = 3h$; $h = 1,2,.....$

Case (3): If $C_n = 3k'-1$; $k = 2,3,....$, then $\{C_{n-2}\} \in F_1$ and $\{C_n, C_{n-1}\} \in F_2$
where $l = 3k'-2$; $l = 2,3,.....$ Clearly $\gamma_R(T) = |F_0 \cup F_1 \cup F_2|$.

Now in $n(T)$, if $C_n = C_3$ and $2k+2$; $k = 1,2,.....$ and $\{e_i\}$ is the set of minimum number of edges incident with F_1 and F_2 or $N(F_2)$, then $\{e_i\}$ belongs to F'_2 in $n(T)$. Suppose $C_n = 2k'+1$; $k' = 2,3,....$ be the minimum number of edges incident with F_2 or $N(F_2)$ in T belongs to F'_2 in $n(T)$ and there exist a vertex $c \in C_{n-2}$ in T such that $c \in F'_1$ in $n(T)$. Clearly $(F'_0\, F'_1 \cup F'_2) = \gamma_{Rn}(T)$. Hence $\gamma_{Rn}(T) \ge \gamma_R(T)$.

Finally we obtain the Nordhaus-Gaddum type result.

Theorem [21]: Let G be a connected (p,q) graph such that both G and $\overline{G}$ are connected, then (i) $\gamma_{Rn}(G) + \overline{\gamma_{Rn}}(G) \le 2\left\lfloor \frac{p}{2} \right\rfloor$.

(ii) $\gamma_{Rn}(G).\overline{\gamma_{Rn}}(G) \le 2\left\lfloor \frac{p}{2} \right\rfloor^2$.

References

1. J. Arquilla and H. Fredricksen, Graphing an Optimal Grsnd Strategy, *Military Operations Research*, *1*, 3-17 (1995).
2. F. Harary, Graph Theory, Adison Wesley, Reading Mass (1972).
3. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graph theory*, Marcel Dekker, New York (1997).
4. Ian Stewart, Defend the Roman Empire!, *Scientific American*, December, 136-138 (1999).
5. V.R. Kulli, Theory of Domination in Graphs, *Vishwa International Publications*, Gulbarga University, Gulbarga, (2010).
6. V. R. Kulli and M. H. Muddebihal, Lict Graph And Litact Graph of a Graph, *Journal of Analysis and Computation*, Vol. *2*, No.1, 33-43 (2006).
7. M. H. Muddebihal, Srinivasa. G. and A. R. Sedamkar, Domination in Square Graphs, *Ultra Scientist* Vol. *23(3) A*, 795-800 (2011).
8. C. S. ReVelle and K. E. Rosing, *Johns Hopkins Review*, April (1997).