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Connected lict domination in graphs

M.H. MUDDEBIHAL and KALSHETTI. SWATI. M.

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Abstract

For any graph G, the lict graph $n(G) = J$ of a graph G is the graph whose vertex set is the union of the set of edges and set of cut vertices of G in which two vertices are adjacent if and only if corresponding members are adjacent or incident. A dominating set D' is called connected dominating set of $n(G)$, if D' is also connected, the minimum cardinality of D' is called connected domination number of $n(G)$ and is denoted by $\gamma_{nc}(G)$. In this paper, many bounds on $\gamma_{nc}(G)$ were obtained in terms of vertices, edges and other different parameters of G but not in terms of elements of J.Further we develop its relation with other different domination parameters.

Key words: Lict Graph / Dominating set / Connected domination/ Total domination.

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1. Introduction

In this paper, all the graphs considered here are simple, finite, non-trivial, undirected and connected. As usual p and q denote, the number of vertices and edges of a graph G. In this paper, for any undefined terms or notations can be found in Harary[4].

The study of domination in graphs was begun by Ore[6] and Berge[2]. The domination in graphs discussed by S.Mitchell and S.T. Hedetineim[5].

As usual, the maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex v is called a cut vertex if removing it from G increases the number of components of G. For any real number x, $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the greatest integer not greater than x. The complement $\overline{G}$ of a graph G has V as its vertex set, but two vertices are adjacent in $\overline{G}$ if they are not adjacent in G. A graph G is called trivial if it has no edges. If G has at least one edge then G is called a non-trivial graph. A non-trivial connected graph G with atleast one cut vertex is called a separable graph, otherwise a non-separable graph.

A vertex cover in a graph G is a set of vertices that covers all the edges of G. The vertex covering number $\alpha_0(G)$ is a minimum cardinality of a vertex cover in G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G. A set of vertices/edges in a graph G is called an independent set if no two vertices/edges in the set are adjacent. The vertex independence number $\beta_0(G)$ is the maximum cardinality of an independent set of vertices. The edge independence number $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges.

The greatest distance between any two vertices of a connected graph G is called the diameter of G and is denoted by $diam(G)$.

We begin by recalling some standard definition from domination theory.

A set D of a graph $G=(V,E)$ is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G.

A dominating set D is a total dominating set if the induced subgraph $\langle D\rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set in G. This concept was introduced by Cochayne, Dawes and Hedetniemi. in[3].

A set F of edges in a graph G is called an edge dominating set of G if every edge in $E-F$ where E is the set of edges in G is adjacent to atleast one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G. The edge dominating set F is called connected edge dominating set of G, if induced subgraph $\langle F\rangle$ is also connected. The connected edge dominating set of G is denoted by $\gamma_c'(G)$ and is the minimum cardinality of connected dominating set. Edge domination number was studied by S.L. Mitchell and Hedetniemi in[5].

A dominating set D is called connected dominating set of G if the induced sub graph $\langle D\rangle$ is connected. The connected domination number $\gamma_c(G)$ of a graph G is the minimum cardinality of a connected dominating set found in[7].

A set $D'\subseteq V'$ is said to be dominating set of $n(G)$, if every vertex in $V'-D'$ is adjacent to some vertex in D'. The domination number of $n(G)$ is denoted by $\gamma_n(G)$ and is the minimum cardinality of dominating set in $n(G)$.

Analogously, we define connected domination number in lict graph as follows.

A dominating set D' of lict graph

$J = n(G)$ is connected dominating set, if induced subgraph $\langle D' \rangle$ is also connected .The connected domination number of $n(G)$ is denoted by $\gamma_{nc}(G)$ and is the minimum cardinality of a connected minimal dominating set in $n(G)$.

In this paper, many bounds on $\gamma_{nc}(G)$ were obtained and expressed in terms of vertices, edges of G but not the elements of $n(G)$. Also we establish connected domination number of a lict graph $n(G)$ and express the results with other different domination parameters of G.

2. Results

We need the following Theorems to establish our further results.

Theorem A[1] *:* For any connected (p,q) graph G, $\gamma'(G) \le \left\lfloor \frac{p}{2} \right\rfloor$.

Theorem B[6] *:* If G is a graph with no isolated vertex, then $\gamma(G) \le \frac{p}{2}$.

Initially we begin with connected domination number of Lict graph of some standard graphs, which are straight forward in following theorem.

Theorem 1:

(i) For any cycle C_p with $p \ge 3$ vertices,

$$\gamma_{nc}(C_p) = p - 2.$$

(ii) For any path P_p with $p \ge 2$ vertices,

$$\gamma_{nc}(P_p) = p - 1 \text{ ; for } p = 2.$$
$$= p - 2 \text{ ; for } p = 3.$$
$$= p - 3 \text{ ; for } p \ge 4.$$

(iii) For any star $K_{1,p}$ with $p \ge 2$ vertices,

$$\gamma_{nc}(K_{1,p}) = 1.$$

(iv) For any wheel W_p with $p \ge 4$ vertices,

$$\gamma_{nc}(W_p) = \frac{p}{2} \text{ ; if } p \text{ is even.}$$
$$= \left\lfloor \frac{p}{2} \right\rfloor \text{ ; if } p \text{ is odd.}$$

In the following Theorem, we obtain the upper bound for $\gamma_{nc}(G)$ in terms of vertices of the G.

Theorem 2. For any connected (p,q) graph G, $\gamma_{nc}(G) \le \left\lceil \frac{p}{2} \right\rceil + 2.$

Proof: Let $F = \{e_1, e_2, e_3, \ldots\ldots, e_n\}$ be an edge dominating set of G and $C = \{c_1, c_2, c_3, \ldots\ldots, c_n\}$ be the set of cutvertices in G. Let $D' = \{v_1, v_2, v_3, \ldots\ldots v_n\}$ be the minimal dominating set of $n(G)$. In $n(G)$, $F \in V[n(G)]$ which is also a dominating set of $n(G)$, such that $F = D'$ and $|D'| = \gamma[n(G)]$. If for some $v_i \in S$ such that $v_i \notin D'$ in $n(G)$, then $D' = F \cup C$. Otherwise $D' = F$. Further let $H = \{u_1, u_2, u_3, \ldots\ldots, u_i\}$ for some $u_i \in V[n(G)]$, such that $H \in N(D')$ and $H \subseteq V[n(G)] - D'$.

Now we consider $H' \subset H$ such that $\langle D' \cup H' \rangle$ is the minimal connected dominating set of $n(G)$. Also by Theorem A, clearly it follows that $|D' \cup H| \leq \left\lceil \frac{p}{2} \right\rceil + 2$ and thus we have

$\gamma_{nc}(G) \leq \left\lceil \frac{p}{2} \right\rceil + 2.$

Theorem 3: For any (p,q) tree T, $\gamma_{nc}(T) \leq p - m$, where m is the number of end vertices in T. Equality holds if $T = K_{1,p}$ with $p \geq 2$ vertices.

Proof: Suppose $diam(T) \geq 3$, then we consider $V_1 = \{v_1, v_2, v_3, ..., v_n\}$ be the set of all end vertices of T, such that $|V_1| = m$. Further suppose $F = \{e_1, e_2, e_3, ..., e_j\}$ be the set of all non-end edges which forms a connected edge dominating set of T. Now without loss of generality, the corresponding edges of F forms a vertex set $D' = \{u_1, u_2, u_3, ..., u_j\}$ in $n(T)$, which is minimal connected dominating set of $n(T)$. Since for any tree T, $q = p - 1$. In $n(T)$, $V[n(T)] = q + C$, where C is the number of cutvertices in T. Now clearly $|D'| \leq p - |V_1|$. Hence $\gamma_{nc}(T) \leq p - m$. Further equality holds if $T = K_{1,p}$, $n[K_{1,p}] = K_{p+1} = 1$ and $\gamma_{nc}(K_{1,p}) = p - m.$

Theorem 4 : For any connected (p,q) graph G, $\gamma_{nc}(G) \geq \left\lceil \frac{diam(G)+1}{3} \right\rceil.$

Proof: Let D' be a connected γ – set of $n(G)$. Consider an arbitrary path of length, which is a $diam(G)$. This diameteral path induces atmost two edges from the induced subgraph $\langle N[v] \rangle$ for each $v \in D'$. Furthermore since D' is a connected γ -set, the diametral path includes atmost $\gamma_{nc}(G) - 1$ edges joining the neighbourhood of the vertices of D'. Hence $diam(G) \leq 2\gamma_{nc}(G) + \gamma_{nc}(G) - 1$.

$$diam(G) \leq 3\gamma_{nc}(G) - 1.$$

Hence the result follows.

The following theorem gives the relation between $\gamma_{nc}(G)$ and $\gamma(G^2)$.

Theorem 5: For any connected (p,q) graph G, $\gamma_{nc}(G) \geq \gamma(G^2)$.

Proof: Let $D' = \{v_1, v_2, v_3, ..., v_n\}$ be a minimal dominating set of $n(G)$, and suppose $D_1 = \{u_1, u_2, u_3, ..., u_n\}$ be the set of vertices in $n(G)$, where $D_1 \subseteq V[n(G)] - D'$ and $D_1 \in N(D')$. Now we consider $D_2 \subset D_1$ such that $\langle D' \cup D_2 \rangle$ is a connected minimal dominating set of $n(G)$. Now without loss of generality in G^2, let $D = \{a_1, a_2, a_3, ..., a_k\}$ is the set of minimal vertices with a maximum degree which are at distance atleast two apart in G^2, covers all the vertices of G^2, then D is a dominating set of G^2. Since the distance between the vertices of D in G^2 is atmost two. Clearly $D \subseteq (D' \cup D_2)$, thus it follows that $|D| \leq |D' \cup D_2|$ and hence $\gamma_{nc}(G) \geq \gamma(G^2)$.

Theorem 6: For any connected (p,q) graph G, $\gamma_{nc}(G) \le 2diam(G) - 2$.

Proof: Let $V = \{v_1, v_2, v_3, ..., v_n\}$ be the set of vertices in G, then there exists two vertices $u, v \in V(G)$, such that dist (u,v) forms a diametral path in G. Now without loss of generality in G. Let $F = \{e_1, e_2, e_3, ..., e_n\}$ be the minimal edge dominating set of a G and $C = \{c_1, c_2, c_3, ...c_k\}$ be the set of cutvertices in G. Further suppose $D' = \{u_1, u_2, u_3, ..., u_j\}$ be the minimal dominating set of $n(G)$, such that $D' \subset F \cup C$. Now let $H = \{a_1, a_2, a_3, ..., a_j\}$ be the set of vertices such that $H \in N\{u_j\}$ of D' and $H \subseteq V[n(G)] - D'$. Further we consider $H' \subset H$ such that $\langle D' \cup H' \rangle$ is the minimally connected dominating set of $n(G)$. Since the set D' includes the diametral path. Hence it is known that $|D' \cup H'| \le 2diam(G) - 2$. Thus it follows that $\gamma_{nc}(G) \le 2diam(G) - 2$.

The following Theorem relates $\gamma_{nc}(G)$ and other domination parameters of G.

Theorem 7 : For any non-trivial connected (p,q) graph G,

$$\gamma_{nc}(G) + \gamma_t(G) \le p + \alpha_0(G).$$

Equality holds, if $G \cong P_2$.

Proof: Suppose that D is a minimal dominating set of G, then $D \cup H$, where $H \in V(G)$ and $H \subseteq V - D$ is a total dominating set of G. Further, let $F = \{e_1, e_2, e_3, ..., e_j\}$ be a minimum independent set of edges in G, which is also an edge dominating set of G. Then the corresponding set $F = D' \subseteq V[n(G)]$ forms a vertex dominating set of $n(G)$. Let $B = \{v_1, v_2, v_3, ..., v_i\}$ be the minimum number of vertices which covers all the edges of G, such that $|B| = \alpha_0$. Let $D_1 = \{u_1, u_2, u_3, ..., u_i\}$ be the vertex set in $n(G)$, such that $D_1 \subseteq V[n(G)] - D'$ and $D_1 \in N(D')$. Now we consider $D_2 \subset D_1$, such that $\langle D' \cup D_2 \rangle$ is the minimal connected dominating set, which gives $|D' \cup D_2| \cup |D \cup H| \le p + |B|$. Thus it follows that $\gamma_{nc}(G) + \gamma_t(G) \le p + \alpha_0(G)$. Equality holds if $G \cong P_2$, in this case G has a unique vertex covering $v \in G$ and $u \in N(v)$ and also $n(P_2) = K_1$ and $\gamma_{nc}(P_2)$ is a trivial, which gives the result.

Theorem 8: For any connected (p,q) graph G, $\gamma_{nc}(G) \le \gamma_t(G) + \left\lceil \frac{p}{2} \right\rceil$.

Proof: Let $D = \{v_1, v_2, v_3, ..., v_n\}$ be a dominating set of G and $V' = V(G) - D$, where $V' \in N(D)$, suppose $H \subseteq V'$ be the minimum set of vertices which are adjacent to D, then $(D \cup H)$ is the total dominating set of G. Further we consider $F = \{e_1, e_2, e_3, ..., e_j\}$ be the minimum set of edges which are incident to the vertices of D forms an edge dominating set of G. Now without loss of generality, let $D' = \{u_1, u_2, u_3, ..., u_n\}$ be the dominating set of $n(G)$. Since $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_3$,.............., $u_n = e_n$, such that $u_i \in V[n(G)]$ and $e_i \in E(G)$, where $1 \le i \le n$. If $V_1 = V[n(G)] - D'$ and $D_1 = \{p_1, p_2, p_3, ..., p_j\}$, where $D_1 \subseteq V_1$ and $D_1 \in N(D')$. Now we consider $D_2 \subset D_1$ then $\langle D' \cup D_2 \rangle$ forms a connected minimal dominating set of $n(G)$. Clearly $|D' \cup D_2| \le |D \cup H| + \left\lceil \frac{p}{2} \right\rceil$ and hence it follows that $\gamma_{nc}(G) \le \gamma_t(G) + \left\lceil \frac{P}{2} \right\rceil$.

The following Theorem gives the relation between the connected domination number of $n(T)$ with connected domination number of a tree T.

Theorem 9: For any tree T, $\gamma_{nc}(T) \le \gamma_c(T)$.

Proof: Suppose a (p,q) graph G is a tree. Let $E = \{e_1, e_2, e_3, ..., e_n\}$ be the set of all non end edges of T and $K = \{u_1, u_2, ..., u_n\}$ be the set of all the cut vertices of T, such that $|K| = \gamma_c(T)$. In $n(T)$, the set $E \in V[n(T)]$. Since for any non trivial (p,q) tree T, $p = q+1$ and hence $|E| = \gamma_{nc}(T)$ gives $\gamma_c(T) \ge \gamma_{nc}(T)$.

Theorem 10: For any non-trivial connected graph G, $\gamma_{nc}(G) \le p - \gamma'(G)$ and $G \ne P_p\,(p > 10), C_p\,(p > 6)$.

Proof: Suppose $V(G) = p$, now let $A = \{e_1, e_2, e_3, ..., e_n\}$ be the edge set which covers all the vertices of G, such that $|A| = \alpha_1$. Let $B = E(G) - A$ be the maximum independent set of edges in G, such that $|B| = \beta_1$. Suppose $F = \{e_1, e_2, e_3, ..., e_j\}$ be the minimum independent edge dominating set. Then F itself is a edge dominating set of G. Further corresponding edges of F forms a vertex set $D' = \{v_1, v_2, v_3, ..., v_i\}; \forall u_i \in n[V(G)]$, which is a dominating set in $n(G)$. If $H = V[n(G)] - D'$ and let $D_1 = \{u_1, u_2, u_3, ...u_i\}$; $\forall u_i \in n[V(G)]$ be the set of vertices of $n(G)$, such that $D_1 \in N(D')$ and $H \subset D_1$ in $n(G)$. Then $\langle D' \cup H \rangle$ is a minimal connected dominating set of $n(G)$. Now $|D' \cup H| \cup |F| \le |A| \cup |B|$. Which gives the result as, $\gamma_{nc}(G) + \gamma'(G) \le \alpha_1 + \beta_1$.

$$\gamma_{nc}(G) + \gamma'(G) \le p.$$

For $G \ne P_p$ with $p > 10$, we assume that graph $G = P_p$ with $p > 10$. Let this path will be denoted as $P: v_1, e_1, v_2, e_2, ..., v_{p-1}, e_{p-1}, v_p$. In $n(G)$, $E: e_1, e_2, e_3, ..., e_p$, gives the induced subgraph, which is a path with $p-1$ vertices. Let D' be a minimal connected dominating set of $n(G)$, such that $|D'| = p-3$, where as $p-\gamma'(G) = p - \left\lceil \frac{P}{2} \right\rceil$. Now one can easily verify that $|D'| > \frac{p}{2}$. Which gives $\gamma_{nc}(G) > p-\gamma'(G)$. Hence $G \neq P_p$ with $p > 10$.

Further if graph $G = C_p$ with $p > 6$. Let C_p be the cycle with $p > 6$ denoted as $C : v_1, e_1, v_2, e_2, ..., v_p, e_p, v_1$. In $n(G)$, $E_1 : e_1, e_2, e_3, ..., e_p$ gives the cycle C_p with same number of vertices. Let D' be the minimal connected dominating set of $n(G)$, such that $|D'| = p-2$, where as $p-\gamma'(G) = p - \left\lceil \frac{p}{3} \right\rceil$.

Now one can easily verify that $|D'| > \frac{2p}{3}$, which gives $\gamma_{nc}(G) > p-\gamma'(G)$. Hence $G \neq C_p$ with $p > 6$.

Theorem 11: For any connected graph G, $\gamma_{nc}(G) \leq p-\Delta(G)$.

Proof: We consider following cases,

Case 1: Suppose G is a tree. Then $E(G) = \{e_1, e_2, e_3, ..., e_n\}$ be the set of all non end edges. Let $|D'| = |E(G)|$ be the vertex set of $n(G)$, which corresponds to the set E and is minimal connected dominating set of $n(G)$. Let there exists a vertex $v_i \in \Delta(G)$ and also by Theorem B, $\gamma(G) \leq \frac{p}{2}$, which gives the result of connected dominating set of $n(G)$, such that $|D'| \leq p-\Delta(G)$ which gives $\gamma_{nc}(G) \leq p-\Delta(G)$.

Case 2: Suppose G is not a tree, again we consider the subcases of case 2.

Subcase 2.1: Assume G is a cycle C_p $(p \geq 3)$, By Theorem 1, $\gamma_{nc}(C_p) = p-2 = p-\Delta(G)$.

Subcase 2.2: Assume G is neither a cycle nor a tree. Let $E = \{e_1, e_2.e_3, ..., e_k\}$ be the connected edge dominating set in G, such that atleast one of $e_i \in \langle G' \rangle; 1 \leq i \leq n$, where G' is a block or a cycle in G. In $n(G)$, the set $E = \{e_1, e_2, e_3, ..., e_k\}$ gives the connected dominating set of $n(G)$. Hence $|E| = \gamma_{nc}(G)$. Suppose there exits $v \in \Delta(G)$ and atleast two edges which are incident to v are the element of E, which gives $p - \Delta(G)$. One can easily verify that $p-\Delta(G) \geq \gamma_{nc}(G)$.

The following result gives the relation between connected domination number of lict graph of G with domination number of lict graph of G.

Theorem 12: For any non-trivial connected (p,q) graph G, $\gamma_{nc}(G) \geq \gamma_n(G)$.

Proof: Let $D' = \{v_1, v_2, v_3, ..., v_n\}$ be a minimal dominating set of $n(G)$, such that $D' \subset F \cup C$, where $F = \{e_1, e_2, e_3, ..., e_k\}$ is the edge dominating set of G and $C = \{c_1, c_2, c_3, ..., c_n\}$ is the set of cutvertices in G. Now let $D_1 = \{u_1, u_2, u_3, ..., u_k\}$ be the set of vertices of $n(G)$, such that $D_1 \subseteq V[n(G)] - D'$ and $D_1 \in N(D')$ in $n(G)$. Further if we consider $D_2 \subset D_1$, then $\langle D' \cup D_2 \rangle$ is the minimal connected dominating set of $n(G)$. Also $D' \subseteq (D' \cup D_2)$. Hence it follows that $|D'| \leq |D' \cup D_2|$. So $\gamma_n(G) \leq$ $\leq \gamma_{nc}(G)$.

Theorem 13: For any non-trivial connected (p,q) graph G, $\gamma_{nc}(G) \leq p - \gamma(G)$; if $G \neq P_p\,(p > 9)$ and $C_p\,(p > 6)$.

Proof: Let $D = \{v_1, v_2, v_3, ..., v_n\}$ be the minimal dominating set of G and $C = \{c_1, c_2, c_3, ..., c_n\}$ be the set of cutvertices of G and let $S = \{e_1, e_2, e_3, ..., e_n\}$ be the set of all non-end edges which are incident to the vertices of D. We consider the set $S_1 = \{e_1, e_2, e_3, ..., e_j\}$; $1 \leq j \leq n$ and $S_1 \subset S$ with the property that $\forall e_j$ is incident to atleast one v_i; $1 \leq v_i \leq n$, $\forall\ v_i \in D$ in G and S_1 is an edge dominating set of G. Now suppose there exists an edge set $S_2 \in E(G)$, where $S_2 = \{e_1, e_2, e_3, ..., e_k\}$ and $\forall e_k$ which are adjacent to any element $e_i \in S_2$ in G. Then $S_1 \cup S_2$ gives the connected edge dominating set in G. In, $n(G)$, these sets S_1 and S_2 preserves the one to one correspondence with $D' \cup D_1$, where D' and D_2 are the correspon-ding vertices of $S_1 \cup S_2$, which gives the result,

$$\gamma_{nc}(G) \leq p - \gamma(G).$$

Suppose $G \neq P_p$ with $p > 9$, then for the graph $G = P_p$ with $p > 9$. Let this path will be denoted as $P: v_1, e_1, v_2, e_2, ..., v_{p-1}, e_{p-1}, v_p$. In $n(G)$, $E: \{e_1, e_2, e_3, ..., e_p\}$ gives the subgraph, which is a path with $p-1$ vertices. Let D' be a minimal connected dominating set of $n(G)$, such that $|D'| = p-3$, where as $p - \gamma(G) = p - \left\lceil \frac{P}{3} \right\rceil$. Now one can easily verify that $|D'| > \frac{2p}{3}$. Which gives $\gamma_{nc}(G) > p - \gamma(G)$. Hence $G \neq P_p$ with $p > 9$.

Further if graph $G = C_p$ with $p > 6$, let C_p be the cycle with $p > 6$, $C: v_1, e_1, v_2, e_2, v_3, e_3, ..., v_p, e_p, v_1$. In $n(G)$: $E_1 = \{e_1, e_2, e_3, ..., e_n\}$ gives cycle C_p with same vertex set of G. Let D' be the minimal connected dominating set of $n(G)$, such that $|D'| = p-2$, where as $p - \gamma(G) = p - \left\lceil \frac{p}{3} \right\rceil$. Now one can easily verify that

$\left|D'\right| > \dfrac{2p}{3}$, which gives $\gamma_{nc}(G) > p - \gamma(G)$. Hence $G \neq C_p$ with $p > 6$.

The following Theorem gives the relation between $\gamma_{nc}(G)$ and $\gamma_c^{'}(G)$.

Theorem 14: For any non-trivial connected (p,q) graph G, $\gamma_{nc}(G) = \gamma_c^{'}(G)$.

Proof: Let $F = \{e_1, e_2, e_3, ..., e_n\}$ be the set of all non-end edges which forms a connected edge dominating set of G, such that $|F| = \gamma_c^{'}(G)$. Now without loss of generality the corresponding edges forms a vertex set $D' = F$ in $n(G)$, which is also a minimal connected dominating set of $n(G)$, such that $|D'| = \gamma_{nc}(G)$. Hence it is clear that $|D'| = |F|$. Thus it follows that $\gamma_{nc}(G) = \gamma_c^{'}(G)$.

Theorem 15: For any non-trivial connected (p,q) graph G, $\gamma_{nc}(G) \leq \gamma_c^{'}(G) + \gamma_n(G)$.

Proof: By using the reference of above Theorem 12 and Theorem 16, we get the result.

Theorem 16: For any non –trivial connected (p,q) graph G, $\gamma_{nc}(G) \geq \gamma'(G)$.

Proof: Let $F = \{e_1, e_2, e_3, ..., e_n\}$ be the minimal edge dominating set of G, such that $|F| = \gamma'(G)$. Let $D' = \{v_1, v_2, v_3, ..., v_n\}$ be the vertex set of $n(G)$, which corresponds to set F and is minimal dominating set of $n(G)$. Further let $D_1 = \{u_1, u_2, u_3, ..., u_k\}$ be the set of vertices in $n(G)$, such that $D_1 \subseteq V[n(G)] - D'$ and $D_1 \in N(D')$. We consider the set D_2 with the property $D_2 \subset D_1$, then $\langle D' \cup D_2 \rangle$ is the minimal connected dominating set of $n(G)$.Hence it is clear that $|D' \cup D_2| \geq |E|$. Thus we get the result $\gamma_{nc}(G) \geq \gamma'(G)$.

Theorem 17: For any connected (p,q) graph G, $\gamma_{nc}(G) \leq \gamma_t(G) + \Delta(G)$.

Proof: Let D be any minimal dominating set of G.Suppose $H \subseteq V$ be the minimum set of vertices which are adjacent to D, then $D \cup H$ is the total dominating set of G. Further let $D' = \{v_1, v_2, v_3, ..., v_n\}$ be the dominating set of $n(G)$ and $D_1 = \{u_1, u_2, u_3, ..., u_k\}$ be the set of vertices of $n(G)$, such that $D_1 \in N(D')$ and we consider $D_2 \subset D_1$, such that $\langle D' \cup D_2 \rangle$ is the minimal connected dominating set of $n(G)$. Now suppose $B=\{u_1,u_2,u_3,.........,u_k\}$ be the set of non end vertices in G, then there exists at least one vertex with the maximum degree $\Delta(G)$ in $B\in G$. Clearly $|D'\cup D_2|\le|D\cup H|$ $+\Delta(G)$. Thus it follows that $\gamma_{nc}(G)\le\gamma_t(G)$ $+\Delta(G)$.

Finally we obtain the Nordhous – Gaddum type result,

Theorem 18: Let G be a connected (p,q) graph such that both G and $\overline{G}$ are connected, then

i) $\gamma_{nc}(G)+\gamma_{nc}(\overline{G})\ge p-3.$

ii) $\gamma_{nc}(G).\gamma_{nc}(\overline{G})\ge\left\lceil\frac{p}{2}\right\rceil.$

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