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The double-ended queue with social and economic situations

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Abstract

The double-ended queue involving trains and passangers at a Railway station has been considered under the assumption that there is limited waiting space both for trains and for passangers. The arrival distribution of trains is simulated by means of an arrival timing channel consisting of j phases. The arrivals of passangers are assumed to be Poisson distributed. In the time dependent case we obtain an expression for the Laplace transform of the probabilities that there are n units in the queue. The cases of exponential and 2-Erlang arrival distributions for trains have been considered as particular cases. In the steady state, for a 2-Erlang arrival distribution for trains, we have evaluated the probabilities that the waiting space for passangers or trains is full in different particular cases. Their values are also shown by means of graphs.

Key words : Double ended queue, limited waiting space, social situations.

1.0 Introduction

Kashyap[3] has discussed the double-ended queue involving queues of taxis and customers at a taxi stand with limited waiting space for both taxis and customers, and the arrivals of customers and taxis both being assumed to be according to the Poisson distribution. Kashyap[4] further extended the result to the case of general arrival distribution for the taxis by using the supplementary variable technique and succeeded in finding the Laplace transform of the state probability generating function in the time dependent case. In this present paper we use the phase technique to study the same queuing process between trains and passengers. This approach has the advantage of lending itself to easy computation of steady state probabilities in the special case of 2-Erlang arrival distribution of trains which was not possible in the previous study[4]. In the particular case of exponential arrival distribution for trains the corresponding result of Kashyap[3] and Jaiswal[2] are shown to follow. For a 2-Erlang arrival distribution, the probabilities that the waiting space for passengers or trains is full, are evaluated for different values of p, N and M, where $\rho = \frac{\lambda}{\mu}$ and N and M denote respectively the maximum number of passengers an trains allowed to wait at the Railway station.

1.1 *Statement of the problem :*

Passengers arrive at a Railway station according to a Poisson distribution with mean arrival rate λ, and depart with a train if they find a queue of trains waiting, or else form a queue. The arrival distribution of trains at the station is simulated by means of an arrival timing channel consisting of j phases, each phase being exponential with mean $\frac{1}{\mu}$, the channel being fed by an infinite source. Phases are numbered in the reverse order. Each time the arrival channel becomes empty, a unit (trains) is put into the channel, starting at the *r*th phase with probability C_r (r=1,2, m...,j). After completing the *r*th phase, it goes to the (r-1)th phase, then to the (r-2)th and so on till the 1st phase, after finishing which it arrives at the station leaving the channel empty. After arriving at the railway station the train departs taking one passengers from the queue, if any, or waits.

There is limited waiting space for M trains or N passengers. Let $P_{n,r}(t)$ be the probability that at time t there are n passengers waiting at the Railway station the arriving trains being in the rth phase. When n is positive, passengers are waiting; when n is zero, neither trains nor passengers are waiting; and when n is negative, it signifies that – n trains are waiting.

2.0 *Formulation of equations :*

Following Keilson and Kooharian[1], we have the following transition equations for the system:

$$P_{-M,r}(t+\Delta)=P_{-M,r}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta P_{-M,r}(t)+\mu\Delta C_r P_{-M+1,1}(t)+\mu\Delta C_r P_{-M,1}(t),[1\le r<j] \tag{2.1}$$

$$P_{-M,j}(t+\Delta)=P_{-M,j}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta C_j P_{-M+1,1}(t)+\mu\Delta C_j P_{-M,1}(t) \tag{2.2}$$

$$P_{n'r}(t+\Delta)=P_{n,r}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta P_{n,r+1}(t)+\mu\Delta C_r P_{n+1,1}(t) + \lambda\Delta P_{n-1,r}(t)\,[-M<n<N,1\le r<j] \tag{2.3}$$

$$P_{n,j}(t+\Delta)=P_{n,j}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta C_j P_{n+1,1}(t)+\lambda\Delta P_{n-1,j}(t), \quad [-M<n<N] \tag{2.4}$$

$$P_{N,r}(t+\Delta)=P_{Nr}(t)[1-\mu\Delta]+\mu\Delta P_{Nr+1}(t)+\lambda\Delta P_{N-1,r}(t),\,[1\le r<j] \tag{2.5}$$

$$P_{N,j}(t+\Delta)=P_{N,j}(t)[1-\mu\Delta]+\lambda\Delta P_{N-1,j}(t) \tag{2.6}$$

Transposing and letting $\Delta \to 0$ we have

$$\frac{d}{dt}P_{-M,r}(t)+(\lambda+\mu)P_{-M,r}(t)=\mu P_{-M,r+1}(t)+\mu C_r P_{-M+1,1}(t)+\mu C_r P_{-M,1}(t) \tag{2.7}$$

$$\frac{d}{dt}P_{-M,r}(t)+(\lambda+\mu)P_{-M,r}(t)=\mu C_j P_{-M+1,1}(t)+\mu C_j P_{-M,1}(t) \tag{2.8}$$

$$\frac{d}{dt}P_{n,r}(t)+(\lambda+\mu)P_{n,r}(t)=\mu P_{n,r+1}(t)+\mu C_r P_{n+1,1}(t)+\lambda P_{n-1,r}(t) \tag{2.9}$$

$$\frac{d}{dt}P_{n,r}(t)+(\lambda+\mu)P_{n,j}(t)=\mu P_{n+1,1}(t)+\lambda P_{n-1,j}(t) \tag{2.10}$$

$$\frac{d}{dt}P_{n,r}(t)+\mu P_{N,r}(t)=\mu P_{N,r+1}(t)=\lambda P_{N-1,r}(t) \tag{2.11}$$

$$\frac{d}{dt}P_{N,j}(t)+\mu P_{N,j}(t)=\lambda P_{N-1,j}(t) \tag{2.12}$$

Now let us define the following generating functions

$$F_r(\alpha,t)=\sum_{n=-M}^{N}\alpha^n P_{n,r}(t) \text{ and } F(\alpha,\beta,t)=\sum_{r=1}^{j}F_r(\alpha,t)\beta^r .$$

Multiplying (2.7) to (2.12) by appropriate powers of α and β and summing over n form $n=-M$ to n= N, and r from r = 1 to r=j we have

$$\frac{d}{dt}F(\alpha,\beta,t)+\left[\lambda(1-\alpha)+\mu\left(1-\frac{1}{\beta}\right)\right]F(\alpha,\beta,t)=\mu F_1(\alpha,t)\{\frac{C(\beta)}{\alpha}-1\}$$

$$+\lambda\alpha^N(1-\alpha)\sum_{r=1}^{j}P_{N,}r(t)\beta^r+\mu\alpha^{-M}P_{-M,1}(t)C(\beta)\left(1-\frac{1}{\alpha}\right) \tag{2.13}$$

where $C(\beta)=\sum_{r=1}^{j}\beta^r C_r$

Further we define $\overline{P}_{n,r}(s)$ to Laplace transform (L.T.) of $P_{n,r}(t)$, viz.,

$$\overline{P}_{n,r}(s)=\int_0^\infty e^{-st}\,P_{n.r}(t)\,dt \qquad (2.14)$$

the transform of other functions being similarly denoted by corresponding letters under a bar, thus for thus for example, $\overline{F}(\alpha,\beta,s)$ is the L.T of $F(\alpha,\beta,t)$.

Let the system start with the arrival of a passenger which makes the number of passenger equal to i and the train in the arrival channel be in the *m*th phase, so we have

$$P_{n,r}(o)=\delta_{t,n.}\,\delta_{m,r.} \qquad (2.15)$$

where $\delta_{t,j}=1,\ \ i=j$

$=0,\ \ i\neq j$

therefore,

$$F(\alpha,\beta,o)=\alpha^{i}.\beta^{m} \qquad (2.16)$$

Now taking L.T. (2.13), we have

$$\left[s+\lambda(1-\alpha)+\mu\left(1-\frac{1}{\beta}\right)\right]\overline{F}(\alpha,\beta,s)=\alpha^{i}.\,\beta^{m}+\lambda\alpha^{N}(1-\alpha)\sum_{r=1}^{j}\overline{P}_{N,r}(s).\beta^{r}$$

$$+\mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)\overline{P}_{-M,1}(s).C(\beta)-\mu\overline{F}_1(\alpha,s)\left\{1-\frac{C(\beta)}{\alpha}\right\}. \qquad (2.17)$$

Since the generating function $F_r(\alpha,t)$ and $F_r(\alpha,\beta,t)$ are finite sums, α and β may be taken as any complex numbers, thus choosing β such that the co-efficient of $\overline{F}(\alpha,\beta,s)$ in the above expression is zero helps to evaluate $\overline{F}_1(\alpha,s)$.

Accordingly now putting $\beta=\dfrac{\mu}{[s+\lambda(1-\alpha)+\mu]}=Z(\text{says})$,

(this amounts to choosing an appropriate β for a given α), we have

$$\overline{F}_1(\alpha,s)=\frac{\left[\alpha^{1}z^{m}+\lambda\alpha^{N}(1-\alpha)\sum_{r=1}^{j}\overline{P}_{N,r}(s)Z^{r}+\mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)C(Z)\overline{P}_{-M,1}(s)\right]}{\mu\left[1-\frac{1}{\alpha}C(Z)\right]} \qquad (2.18)$$

$$C(Z)=\sum_{r=1}^{j}C_r.Z^{r}.$$

The denominator is a polynomial of degree (j+l) in a α and therefore, has j+l zeros. By Rouche's theorem, it can be seen that this equation has j roots inside and one outside the unit circle. Since the expression on the L.H.S. (2.18) is regular over the entire complex plane the numerator must vanish at all the j+l zeros of the denominator, giving rise to j+l equations in the j+l unknowns, viz.,

$$\overline{P}_{-M,1}(s) \text{ and } \overline{P}_{N,}r(s), \quad r = 1,2,\ldots, j.$$

Putting $\beta = 1$ in eqn. (2.16) we get

$$\overline{F}(\alpha, l, s) = \frac{\alpha^{i+1} + (1-\alpha)\left[\lambda\alpha^{N+1}\sum_{r=1}^{j} \overline{P}_{N,r}(s) - \mu\alpha^{-M}.\overline{P}_{-M,1}(s) + \mu\overline{F}_1(\alpha, s)\right]}{\alpha\,[s + \lambda(1-\alpha)]} \tag{2.19}$$

It gives the generating function of the L.T. of the probabilities that there are n units present in the queue. Equation (2.18) and (j+1) equations completely determine (2.19). We are justified here in taking value of $\overline{F}_1(\alpha, s)$ from (2.18) since it is independent of β.

3.0 *Particular Cases :*

3.1 *Exponential Arrival Time Distribution:*

In this case $C_r = \delta_{r,1}$ and m=1, therefore, the zeros of the denominator are given by $\lambda\alpha^2 - (s + \lambda + \mu)\alpha + \mu = 0$, which gives two distinct roots, let these be α_1, and α_2. So, now from eqn. (2.18) we have

$$\alpha_1^{-M}.\ \overline{P}_{\_M,1}(s) - \frac{\lambda}{\mu}\alpha_1^{N+1}, \overline{P}_{N,1}(s) + \frac{\alpha_1^{i+1}}{\mu(\alpha_2 - 1)} = 0 \tag{3.1.1}$$

$$\alpha_2^{-M}.\ \overline{P}_{\_M,1}(s) - \frac{\lambda}{\mu}\alpha_2^{N+2}, \overline{P}_{N,1}(s) + \frac{\alpha_2^{i+1}}{\mu(\alpha_2 - 1)} = 0 \tag{3.1.2}$$

Hence

$$\overline{P}_{N,1}(s) = \frac{\left[\alpha_1^{i+1}.\ \alpha_2^{-M}(1-\alpha_2) - \alpha_2^{i+1}.\alpha_1^{-M}(1-\alpha_1)\right]}{s\left[\alpha_1^{N+1}.\ \alpha_2^{-M} - \alpha_2^{N+1}.\alpha_1^{-M}\right]} \tag{3.1.3}$$

$$\frac{\mu}{\lambda},\ \overline{P}_{-M,1}(s) = \frac{(\alpha_1.\alpha_2)^{i+1}\left[(\alpha_2^{N-i+1} - \alpha_1^{N-i+1}) - (\alpha_2^{N-i} - \alpha_1^{N-i})\right]}{s\left[\alpha_2^{N+1}.\ \alpha_1^{-M} - \alpha_1^{N+1}.\alpha_2^{-M}\right]} \tag{3.1.4}$$

Eqns. (3.1.3) and (3.1.4) agree with corresponding results of Kashyap[3,4,5] and in the particular case M=0, these reduces to similar results of Jaiswal[2].

3.2 *2-Erlang Arrival Time Distribution :*

In this case $C_r = \delta_{r,2}$ and $m=2$, so we have from (2.18)

$$\alpha_1^{N+1}\left[\frac{1}{Z_1}\bar{P}_{N,1}(s)+\bar{P}_{N,2}(s)\right]-\frac{\mu}{\lambda}\alpha_1^{-M}.\bar{P}_{-M,1}(s)+\frac{\alpha_1^{i+1}}{\lambda(1-\alpha_1)}=0 \tag{3.2.1}$$

$$\alpha_2^{N+1}\left[\frac{1}{Z_2}\bar{P}_{N,1}(s)+\bar{P}_{N,2}(s)\right]-\frac{\mu}{\lambda}\alpha_2^{-M}.\bar{P}_{-M,1}(s)+\frac{\alpha_2^{i+1}}{\lambda(1-\alpha_2)}=0 \tag{3.2.2}$$

$$\alpha_3^{N+1}\left[\frac{1}{Z_3}\bar{P}_{N,1}(s)+\bar{P}_{N,2}(s)\right]-\frac{\mu}{\lambda}\alpha_2^{-M}.\bar{P}_{-M,1}(s)+\frac{\alpha_3^{i+1}}{\lambda(1-\alpha_3)}=0 \tag{3.2.3}$$

where α_r, $r=1,2,3$ are the root of

$$1-\frac{1}{\alpha}\left[\frac{\mu}{s+\lambda(1-\alpha)+\mu}\right]^2=0.$$

On solving (3.2.1) (3.2.2) and (3.2.3) we have

$$\bar{P}_{N,1}(s)=\frac{\sum\frac{\alpha_1^{i+1}(\alpha_2.\alpha_3)^{N+1}}{\lambda(1-\alpha_1)},\left[\alpha_3^{-(M+N+1)}-\alpha_2^{-(M+N+1)}\right]}{(\alpha_1.\alpha_2.\alpha_3)^{N+1}\sum\alpha_1^{-(M+N+1)}\left[\frac{1}{Z_3}-\frac{1}{Z_2}\right]} \tag{3.2.4}$$

$$\frac{\mu}{\lambda}.P_{-M,1}(s)=\frac{\sum\frac{\alpha_1^{i-N}}{\lambda(1-\alpha_1)}\left[\frac{1}{Z_3}-\frac{1}{Z_2}\right]}{\sum\alpha_1^{-(M+N+1)}\left[\frac{1}{Z_3}-\frac{1}{Z_2}\right]} \tag{3.2.5}$$

$$\bar{P}_{N,2}(s) = \frac{\alpha_1^{-(M+N+1)} . \sum \frac{\alpha_1^{i-X}}{\lambda(1-\alpha)} \left[\frac{1}{Z_3} - \frac{1}{Z_2} \right]}{\sum \alpha_1^{-(M+N+1)} \left[\frac{1}{Z_3} - \frac{1}{Z_2} \right]} - \frac{\alpha_1^{i-N}}{\lambda(1-\alpha_1)}$$

$$- \frac{1}{Z_1} . \frac{. \sum \frac{\sum \alpha_1^{i+1} (\alpha_2 - \alpha_3)^{N+1}}{\lambda(1-\alpha_1)} \left[\alpha_s^{-(M+N+1)} - \alpha_2^{-(M+N+1)} \right]}{(\alpha_1 . \alpha_2 . \alpha_3)^{N+1} \sum \alpha_1^{-(M+N+1)} \left[\frac{1}{Z_3} - \frac{1}{Z_2} \right]} \tag{3.2.6}$$

where summation is carried over the roots $\alpha_1, \alpha_2, \alpha_3$ and Z_1, Z_2, Z_3, where

$$Z_i = \frac{\mu}{[s + \lambda(1-\alpha) + \mu]}, \ i = 1,2,3.$$

Hence in this case

$$\bar{F} i(\alpha,1,s) = \frac{1}{\alpha[s + \lambda(1-\alpha)]} \left[\alpha^{i+1} + (1-\alpha) \left\{ \lambda \alpha^{N+1} \sum_{r=1}^{2} \bar{P}_{N,r}(s) - \mu \alpha^{-M} . \bar{P}_{-M,1}(s) \right. \right.$$

$$\left. \left. + \frac{\alpha^i Z^2 + \lambda \alpha^N (1-\alpha) \sum_{r=1}^{2} \bar{P}_{N,r}(s) Z^r + \mu \alpha^{-M} \left(1 - \frac{1}{\alpha} \right) Z^2 \bar{P}_{-M,1}(s)}{\left[1 - \frac{Z^2}{\alpha} \right]} \right\} \right] \tag{3.2.7}$$

where $\bar{P}_{N,1}(s), \bar{P}_{N,2}(s)$ and $\bar{P}_{-M,1}(s)$ are given by (3.2.4), (3.2.6) and (3.2.5).

4.0 *Steady state solution :*

For the steady state case we have the well known property.

$$\lim_{s \to 0} s \, \bar{P}_{n,r}(s) = \lim_{t \to \infty} P_{n,r}(t) = P_{n,r}. \tag{4.1}$$

Now applying (4.1) to (2.18) and (2.19) we have

$$F_1(\alpha) = \sum_{n=-M}^{N} \alpha^n . P_{n,1}$$

$$= \frac{\lambda\alpha^N(1-\alpha)\sum_{r=1}^{j} P_{N,r}\left[\frac{\mu}{\lambda(1-\alpha)+\mu}\right]^r + \mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)P_{-M,1}\sum_{r=1}^{j} C_r\left[\frac{\mu}{\lambda(1-\alpha)+\mu}\right]^r}{\mu\left[1-\frac{1}{\alpha}\sum_{r=1}^{j} C_r\left[\frac{\mu}{\lambda(1-\alpha)+\mu}\right]^r\right]} \quad (4.2)$$

and

$$F(\alpha,1) = \left[\alpha^N \sum_{r=1}^{j} P_{N,r} - \frac{\mu}{\lambda}\alpha^{-(M+1)} P_{-M,1} + \frac{\mu}{\lambda\alpha} F_1(\alpha)\right]. \quad (4.3)$$

Now we restrict our discussion for the 2-Erlang arrival time distribution, for which we have from (4.2) and (4.3)

$$F_1(\alpha) = \sum_{n=-M}^{N} P_{n,1}\,\alpha^n$$

$$= \frac{\lambda\alpha^N(1-\alpha)\sum_{r=1}^{2} P_{N,r}\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^r + \mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^2 . P_{M,1}}{\mu\left[1-\frac{1}{\alpha}\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^2\right]} \quad (4.4)$$

and

$$F(\alpha,1) = \left[\alpha^N \sum_{r=1}^{2} P_{N,r} - \frac{\mu}{\lambda}\alpha^{-(M+1)} P_{-M,1} + \frac{\mu}{\lambda\alpha} F_1(\alpha)\right]. \quad (4.5)$$

where $F_1(\alpha)$ is given by (4.4) and $P_{N,1,}$ $P_{N,2}$ and $P_{-N,1}$ are to be determine by making $s \to 0$ in eqns.(3.2.4), (3.2.6), (3.2.5). We notice that now $\alpha_{1,}\,\alpha_{2,}\,\alpha_{3,}$ are the roots of

$$1-\frac{1}{\alpha}\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^2 = 0.$$

Here it is to be noted that $\alpha = 1$ (say α_1) is a root of the above equation, then α_2 and α_3 are the roots of

$$\alpha^2 - \left(1 + \frac{2}{\rho}\right)\alpha + \frac{1}{\rho^2} = 0$$

where $\rho = \lambda / \mu$. Thus now solving (3.2.3) (3.2.4) and (3.2.5)

$$P_{\_M,1} = \frac{1}{2} \frac{(1 - 2/\rho)[\alpha_2 - \alpha_3]}{\sum \alpha_1^{-(M+N+1)}[\alpha_2 - \alpha_3]} \quad (4.6)$$

$$P_{N,1} = \frac{1}{2} \frac{\frac{(1-2\rho)}{\rho^2}\left[\alpha_2^{-(M+N+1)} - \alpha_3^{-(M+N+1)}\right]}{\sum \alpha_1^{-(M+Y+1)}[\alpha_2 - \alpha_3]} \quad (4.7)$$

$$P_{N,2} = \left[\left\{\frac{1}{\rho} P_{-M,1} + 1\right\} - \left\{P_{N,1} + \frac{1}{2\rho}\right\}\right] \quad (4.8)$$

It may be noted that $P_{\_M,1}$ and $P_{N,1}$ are the probabilities that the waiting space respectively for trains and passangers is full when a train is about to arrive at theRailway station.

Also $P_{N,1} + P_{N,2} = P_N$ gives the probability that the waiting space for passanger is full irrespective of the phase of the arrival channel[6,7].

5.0 **Numerical Results**

In this section, we tabulate the values of $P_{N,1}$, $P_{\_M,1}$ for different values of ρ, N and $M(=5)$ for 2-Erlang arrival distribution for trans (Table-I). These values are also shown in Figs. 1 and 2.

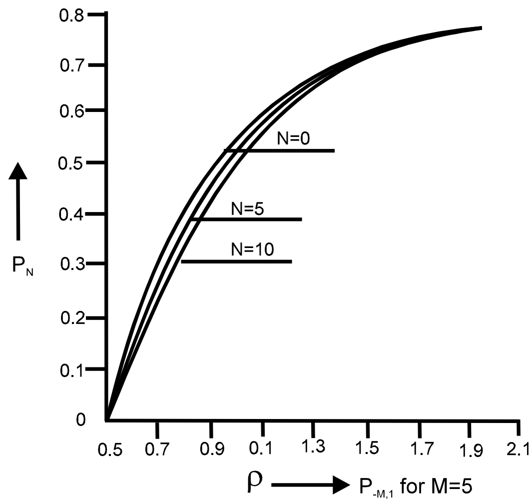

Fig. 1

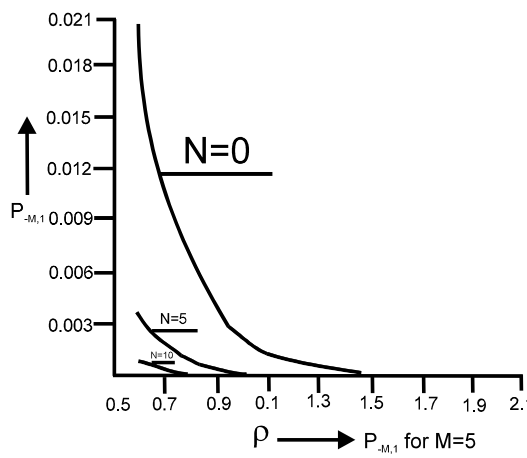

Fig. 2

P/N	$P_{-M,1}$			$P_{N,1}$			$P_{N,2}$		
	0	5	10	0	5	10	0	5	10
0.5	0	0	0	0	0	0	0	0	0
0.6	0.02013	0.003605	0.00015	0.1339	0.1147	0.1112	0.069295	0.057809	0.05655
0.7	0.01621	0.001545	0.000012	0.1984	0.1848	0.1834	0.110267	0.102907	0.10211
0.8	0.0079	0.000273	0.000001	0.2436	0.2377	0.2374	0.141275	0.13764	0.137601
0.9	0.00408	0.000058	0.0000001	0.2808	0.2778	0.27771	0.16843	0.16729	0.16674
1.0	0.00225	0.000016	0.0000000	0.3101	0.30903	0.30902	0.19215	0.19098	0.19058
1.1	0.00120	0.000005	0.0000000	0.3347	0.3386	0.33380	0.21184	0.21159	0.21126
1.2	0.00065	0.000001	0.0000000	0.3540	0.3538	0.35371	0.22998	0.22955	0.22954
1.3	0.00031	0.000000	0.0000000	0.37290	0.3728	0.37273	0.24273	0.24260	0.24260
1.4	0.00014	0.000000	0.0000000	0.37946	0.3793	0.37918	0.26340	0.26355	0.26367
1.5	0.000006	0.000000	0.0000000	0.39509	0.39507	0.39506	0.271564	0.27158	0.37159
1.6	0.000000	0.000000	0.0000000	0.4056	0.4052	0.4049	0.2819	0.2823	0.2826
1.7	0.000000	0.000000	0.000000	0.4138	0.4135	0.41340	0.2921	0.2924	0.2925
1.8	0.000000	0.000000	0.000000	0.4164	0.4162	0.4160	0.3058	0.3060	0.30620
1.9	0.000000	0.000000	0.000000	0.42726	0.42724	0.42722	0.30959	0.30961	0.30963
2.0	0.000000	0.000000	0.000000	0.43320	0.43310	0.43302	0.31680	0.31690	0.316980

6.0 References

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