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# Approximation of conjugate series of a fourier series by product summability

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## Abstract

In this paper a theorem on degree of approximation by product summability $(E,q)(N,p_n)$ of the conjugate series of the Fourier series of the function $f$ of class $Lip(\xi(t),r)$.


*Key words:* Degree of Approximation, $Lip(\alpha,r)$ class of function, $Lip(\xi(t),r)$ class of function, $(E,q)$- mean, $(N,p_n)$- mean, $(E,q)(N,p_n)$-mean, Fourier series, conjugate of the Fourier series, Lebesgue integral.

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## 1. Introduction

**L**et $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty, \text{as } n \to \infty, (P_{-i} = p_{-i} = 0, i \geq 0). \tag{1.1}$$

The sequence –to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_{n-\upsilon} s_\upsilon , \tag{1.2}$$

defines the sequence $\{t_n\}$ of the $(N, p_n)$- mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \to s \quad \text{, as} \quad n \to \infty, \tag{1.3}$$

then the series $\sum a_n$ is said to be $(N, p_n)$ summable to $s$.

The conditions for regularity of $(N, p_n)$-summability are easily seen[1] to be

$$\begin{cases} (i) & \dfrac{p_n}{P_n} \to 0, \text{ as } n \to \infty \\ (ii) & \sum_{k=0}^{n} p_k = O(P_n), \text{ as } n \to \infty \end{cases} \tag{1.4}$$

The sequence–to-sequence transformation[1],

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \quad , \tag{1.5}$$

defines the sequence $\{T_n\}$ of the $(E, q)$ mean of the sequence $\{s_n\}$.

If

$$T_n \to s \text{ , as } \quad n \to \infty, \tag{1.6}$$

then the series $\sum a_n$ is said to be $(E,q)$ summable to $s$.

Clearly $(E,q)$ method is regular. Further, the $(E,q)$ transform of the $(N, p_n)$ transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} T_k$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_\upsilon s_\upsilon \right\} \tag{1.7}$$

If

$$\tau_n \to s \quad \text{, as} \quad n \to \infty, \tag{1.8}$$

then $\sum a_n$ is said to be $(E,q)$ $(N, p_n)$-summable to $s$.

Let $f(t)$ be a periodic function with period $2\pi$ and L-integrable over $(-\pi,\pi)$. The Fourier series associated with $f$ at any point "x" is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \ , \tag{1.9}$$

and the conjugate series of the Fourier series (1.9) is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \tag{1.10}$$

Let $\bar{s}_n(f;x)$ be the n-th partial sum of (1.10). The $L_\infty$-norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\} \tag{1.11}$$

and the $L_\upsilon$ -norm is defined[2] by

$$\left\| f \right\|_{\upsilon} = \left( \int_0^{2\pi} \left| f(x) \right|^{\upsilon} \right)^{\frac{1}{\upsilon}}, \ \upsilon \geq 1. \qquad (1.12)$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\| \, . \, \|_{\infty}$ is defined by [5].

$$\|P_n - f\|_{\infty} = \sup\{|p_n(x) - f(x)| : x \in R\} \qquad (1.13)$$

and the degree of approximation $E_n(f)$ of a function $f \in L_{\upsilon}$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_{\upsilon}. \qquad (1.14)$$

A function $f$ is said to satisfy Lipschitz condition (here after we write $f \in Lip\alpha$) if

$$|f(x+t) - f(x)| = O(|t|^{\alpha}), 0 < \alpha \leq 1, \qquad (1.15)$$

and $f(x) \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right),$$

$$0 < \alpha \leq 1, \ r \geq 1, \ t > 0. \qquad (1.16)$$

For a given positive increasing function $\xi(t)$, the function $f(x) \in Lip(\xi(t), r)$, if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O(\xi(t)),$$

$$r \geq 1, \ t > 0 \qquad (1.17)$$

We use the following notation throughout this paper:

We use the following notation throughout this paper :

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \qquad (1.18)$$

and

$$\overline{K}_n(t) = \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}. \qquad (1.19)$$

Further, the method $(E,q)(N,p_n)$ is assumed to be regular.

*2. Known Theorem:*

Dealing with The degree of approximation by the product mean Misra *et al.*[2] proved the following theorem using $(E,q)$ $(\overline{N},p_n)$ mean of conjugate series of Fourier series :

*Theorem 2.1:*

If *f* is a $2\pi -$ periodic function of class $Lip\alpha$, then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means of the conjugate series (1.10) of the Fourier series (1.9) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$ $, 0<\alpha<1$

where $\tau_n$ is as defined in (1.7).

Very recently Paikray *et al.*[3] established a theorem on degree of approximation by the product mean $(E,q)(\overline{N},p_n)$ of the conjugate series of Fourier series of a function of class $Lip\ (\alpha,r)$. They prove:

*Theorem 2.2:*

If ***f*** is a 2π– Periodic function of class $Lip(\alpha,r)$, then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means of on he conjugate series (1.10) of the Fourier series (1.9) is given by

$$\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{r}}}\right),\ 0<\alpha<1,\ r\geq 1,$$

where is as defined in (1.7).

*3. Main theorem:*

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)(N,p_n)$ of the conjugate series of the Fourier series of a function of class $Lip\ (\xi(t),r)$. We prove:

*Theorem 3.1:*

Let $\xi(t)$ be a positive increasing function and $f$ a 2π– Periodic function of the class $Lip(\xi(t),r),\ r\geq 1, t>0$. Then degree of approximation by the product $(E,q)(N,p_n)$ summability means on the conjugate series (1.10) of the Fourier series (1.9) is given by

$$\|\tau_n - f\|_\infty = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right),\ r\geq 1.$$

where $\tau_n$ is as defined in (1.7).

*4. Required Lemmas:*

We require the following Lemmas to prove the theorem.

*Lemma 4.1:*

$$\left|\overline{K}_n(t)\right| = O(n) \quad ,0\leq t\leq \frac{1}{n+1}\ .$$

*Proof:*

For $0\leq t\leq \frac{1}{n+1}$, we have $\sin nt \leq n\sin t$ then

$$\left|\overline{K}_n(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\left(\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\left(O\left(2\sin\upsilon\frac{t}{2}\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\left(O(\upsilon)+O(\upsilon)\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{O(k)}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\right|$$

$$= O(n)$$

This proves the lemma.

*Lemma-4.2:*

$$\left|\bar{K}_n(t)\right| = O\left(\frac{1}{t}\right), \ for \ \ \frac{1}{n+1}\leq t\leq\pi .$$

*Proof:*

For $\frac{1}{n+1}\leq t\leq\pi$, by Jordan's lemma, we have $\sin\left(\frac{t}{2}\right)\geq\frac{t}{\pi}$.

Then

$$\left|\bar{K}_n(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$=\frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}.\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le\frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi}{2t}p_{k-\upsilon}\left(\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}\right)\right\}\right|$$

$$\le\frac{\pi}{2\pi(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\right\}\right|$$

$$=\frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\right\}\right|$$

$$=\frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$=O\left(\frac{1}{t}\right).$$

This proves the lemma.

*5. Proof of theorem- 3.1:*

Using Riemann – Lebesgue theorem, we have for the n-th partial sum $\overline{s_n}(f;x)$ of the conjugate Fourier series (1.10) of $f(x)$, following Titchmarch[4]

$$\overline{s_n}(f;x)-f(x)=\frac{2}{\pi}\int_0^{\pi}\psi(t)\,\overline{K_n}\,dt,$$

the $(N,p_n)$ transform of $\overline{s_n}(f;x)$ using (1.2) is given by

$$t_n-f(x)=\frac{2}{\pi P_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}p_k\frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt,$$

denoting the $(E,q)(N,p_n)$ transform of $\overline{s_n}(f;x)$ by $\tau_n$, we have[5]

$$\|\tau_n-f\|=\frac{2}{\pi(1+q)^n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}\frac{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\}dt$$

$$= \int_0^{\pi} \psi(t)\ \overline{K_n}(t)\, dt$$

$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right\} \psi(t)\ \overline{K_n}(t)\ dt$$

$$= I_1 + I_2\,,\ say \tag{5.1}$$

Now

$$|I_1| = \frac{2}{\pi(1+q)^n} \left| \int_0^{1/n+1} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{2\sin\frac{t}{2}} \right\} dt \right|$$

$$\le \left| \int_0^{\frac{1}{n+1}} \psi(t)\ \overline{K_n}(t)\, dt \right|$$

$$= \left( \int_0^{\frac{1}{n+1}} \left( \frac{\phi(t)}{\xi(t)} \right)^r dt \right)^{\frac{1}{r}} \left( \int_0^{\frac{1}{n+1}} \left( \xi(t) \bar{K}_n(t) \right)^s dt \right)^{\frac{1}{s}}$$ , using Holder's inequality

$$= O(1) \left( \int_0^{\frac{1}{n+1}} \xi(t) n^s dt \right)^{\frac{1}{s}} \qquad = O\left( \xi\left( \frac{1}{n+1} \right) \frac{1}{(n+1)^{\frac{1}{s}-1}} \right).$$

$$= O\left( \xi\left( \frac{1}{n+1} \right) \right) \left( \frac{n^s}{n+1} \right)^{\frac{1}{s}} \qquad = O\left( \xi\left( \frac{1}{n+1} \right) \frac{1}{(n+1)^{-\frac{1}{r}}} \right)$$

$$= O\left( (n+1)^{\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right) \quad (5.2)$$

Next

$$\left| I_2 \right| \leq \left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{\phi(t)}{\xi(t)} \right)^r dt \right)^{\frac{1}{r}}$$

$$\left( \int_{\frac{1}{n+1}}^{\pi} \left( \xi(t) \bar{K}_n(t) \right)^s dt \right)^{\frac{1}{s}},$$

using Holder's inequality

$$= O(1) \left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{\xi(t)}{t} \right)^s dt \right)^{\frac{1}{s}}, \text{using Lemma 4.1}$$

$$= O(1) \left( \int_{\frac{1}{\pi}}^{n+1} \left( \frac{\xi\left(\frac{1}{y}\right)}{\frac{1}{y}} \right)^s \frac{dy}{y^2} \right)^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function, so is $\xi(1/y)/(1/y)$. Using second mean value theorem we get

$$= O\left( (n+1) \xi\left( \frac{1}{n+1} \right) \right) \left( \int_{\delta}^{n+1} \frac{dy}{y^2} \right)^{\frac{1}{s}}, \textit{ for some }$$

$$\frac{1}{\pi} \leq \delta \leq n+1$$

$$= O\left( (n+1)^{\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right)$$

Then from (5.2) and (5.3) , we have

$$\left| \tau_n - f(x) \right| = O\left( (n+1)^{\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right), \text{for } r \geq 1.$$

$$\left\| \tau_n - f(x) \right\|_{\infty} = \sup_{-\pi < x < \pi} \left| \tau_n - f(x) \right|$$

$$= O\left( (n+1)^{\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right), \; r \geq 1.$$

This completes the proof of the theorem.

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