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Properties of Onto Functionally Open Equal Closed Spaces

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Abstract

Within this paper properties of topological spaces for which open, onto functions and closed, onto functions are equivalent are investigated and the results are used to further characterize such spaces.

Key words: onto, open functions, onto, closed functions.

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Introduction

1. Questions and Direction: Within a summer 2007 graduate level topology class, the students were asked to prove or disprove the open image of a T_0 topological space is T_0. The students believed the statement to be false and repeatedly, without success, attempted to use a finite T_0 space to create a counterexample. Thus the following question arose. "Is the open image of a finite T_0 space T_0?" The answer proved to be "yes."

Theorem 1.1. Let X be a nonempty set. Then X is finite iff for each topology T on X for which (X,T) is T_0, each open image of (X,T) is T_0 [1].

In a similar manner, open images and other separation axioms have been used to obtain many additional topological characterizations of nonempty, finite sets [2].

In February 2012, the focus of the study moved from open images to closed images and many additional topological characterizations of nonempty, finite sets were obtained using closed images and separation axioms[3].

By the nature of the initial questions, all topologies on a set satisfying certain properties had to be considered. In a follow-up paper to the paper cited immediately above, the focus moved from all topologies satisfying certain properties to just one topology satisfying certain properties and the following result was obtained.

Theorem 1.2. Let (X,T) be a space. Then T= P(X), the power set of X, iff (X,T) is T_1 and an onto function f from (X,T) to a space

(Y,S) is open iff it is closed[4].

Also, in the follow-up paper[4], the R_0 separation axiom was used to further examine the new focus.

Definition 1.1. A space (X,T) is R_0 iff for each closed set C in X and each $x \notin C$, $C \cap Cl(\{x\}) = \phi$ [9].

Theorem 1.3. Let (X,T) be a space. Then the following are equivalent: (a) T = P(X) or T is the indiscrete topology on X and (b) (X,T) is R_0 and an onto function f from (X,T) to a space (Y,S) is open iff it is closed[4].

The results above raised questions about other spaces for which an open, onto function equals a closed, onto function and led to the introduction and investigation of onto functionally open equal closed spaces[5].

Definition 1.2. A space (X,T) is onto functionally open equal closed (ofoec) iff for each space (Y,S) for which there is a function f from (X,T) onto (Y,S), f is open iff it is closed[5].

The introduction of ofoec spaces raised questions about the properties of such spaces. For example: "What about subspaces of ofoec spaces?" "What about image of ofoec spaces?" "What about product spaces of ofoec spaces?" These questions and others will be addressed within this paper and additional characterizations of ofoec spaces will be given.

2. Properties of OFOEC Spaces. If for a space (X,T) for which an open, onto function is equivalent to a closed, onto function, must the same be true for a nonempty subset Y of X with the subspace topology T_Y? With only the definition of ofoec spaces such a question is overwhelming. Fortunately, within the paper cited above in which ofoec spaces were defined[5], a precise, descriptive characterization of ofoec spaces was obtained, making the task concerning subspaces doable.

Theorem 2.1. A space (X,T) is ofoec iff T= P(X) or T is the indiscrete topology on X or X has two distinct elements x and y and $T= \{\phi, X, \{x\}\}$ or $T= \{\phi, X, \{y\}\}$ [5].

Theorem 2.2. Let (X,T) be a space. Then (X,T) is an ofoec space iff each nonempty subspace (Y,T_Y) is ofoec.

Proof: Suppose (X,T) is ofoec. Let Y be a nonempty subset of X. If T= P(X), then T_Y= P(Y) and (Y,T_Y) is ofoec. If T is the indiscrete topology on X, then T_Y is the indiscrete topology on Y and (Y,T_Y) is ofoec. If X= {x,y}, x not y, and $T= \{\phi, X, \{x\}\}$, then Y= X or Y is a singleton set and, in either case (Y,T_Y) is ofoec.

The converse follows immediately since (X,T) is a subspace of itself.

Combining Theorem 2.2 with the definition of ofoec spaces gives the next result.

Corollary 2.1. A space (X,T) is ofoec iff for each subspace (Y,T_Y) of (X,T), for each function f from (Y,T_Y) onto a space (Z,W), f is open iff it is closed.

Theorem 2.3. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is ofoec, (b) T is the indiscrete topology on X or each open image (Y,S) of (X,T) is ofoec, and (c) T is the indiscrete topology on X or each closed image (Y,S) of (X,T) is ofoec.

Proof: (a) implies (b): If T is the indiscrete topology on X, then (a) implies (b). Thus consider the case that T is not the indiscrete topology on X. If T= P(X), then for each open image (Y,S) of (X,T), singleton sets are open in (Y,S), which implies S= P(Y) and (Y,S) is ofoec. Thus consider the case where X={x,y}, where x and y are distinct, and T= { ϕ,X,{x}}.

Let (Y,S) be an open image of (X,T). If Y is a singleton set, then (Y,S) is ofoec. Thus consider the case Y is a doubleton set. Let f be an open function from (X,T) onto (Y,S). Then S={ϕ,Y,{f(x)}}or S={ϕ,Y,{f(x)}, {f(y)}} and, in either case (Y,S) is ofoec.

(b) implies (c): If T is the indiscrete topology on X, then (b) implies (c). Thus consider the case that T is not the indiscrete topology on X. Let (Y,S) be a closed image of (X,T). Since the identity function from (X,T) onto itself is open, then (X,T) is ofoec. Since (Y,S) is a closed image of (X,T), (Y,S) is an open image of (X,T) and (Y,S) is ofoec.

(c) implies (a): If T is the indiscrete topology on X, then (X,T) is ofoec. Thus consider the case that T is not the indiscrete topology on X. Since the identity function from (X,T) onto itself is closed and onto, (X,T) is ofoec.

Theorem 2.4. Let (X,T) be a space. Then (a) (X,T) is ofoec iff (b) for each open function f from (X,T) into a space (Y,S), the function f from (X,T) onto (f(X),$S_{f(X)}$) is open, closed, and for each closed function g from (X,T) into a space (Z,W), the function g from (X,T) onto (g(X),$W_{g(X)}$) is open, closed.

Proof: (a) implies (b): Let f be an open function from (X,T) into a space (Y,S). Then f is an open function from (X,T) onto (f(X),$S_{f(X)}$) and, since (X,T) is ofoec, f , a function from (X,T) onto (f(X),$S_{f(X)}$), is open and closed.

In a similar manner, the remainder of (b) can be proven.

(b) implies (a): If f is an open function from (X,T) onto a space (Y,S), then f(X)= Y and $S_{f(X)}$= S, which implies f is open and closed. Similarly, if g is a closed function from (X,T) onto (Z,W), g is closed and open. Hence (X,T) is ofoec.

Theorem 2.5. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is ofoec, (b) T is the indiscrete topology on X or for each space (Y,S) for which there is an open function from (X,T) into (Y,S), (f(X),$S_{f(X)}$) is an open ofoec subspace of (Y,S), and (c) T is the indiscrete topology on X or for each space (Z,W) for which there is a closed function g from (X,T) into (Z,W), (g(X),$W_{g(X)}$) is a closed ofoec subspace of (Z,W).

Proof: (a) implies (b): If T is the indiscrete topology on X, (b) is proven. Thus consider the case that T is not the indiscrete topology on X.

Let (Y,S) be a space for which there is an open function from (X,T) into (Y,S). Then f(X) is open in (Y,S) and $(f(X),S_{f(X)})$ is an open image of the ofoec space (X,T), which by Theorem 2.3 implies $(f(X),S_{f(X)})$ is ofoec.

(b) implies (c): If T is the indiscrete topology on X, then (b) implies (c). Thus consider the case that T is not the indiscrete topology on X.

Let (Z,W) be a space for which there is a closed function g from (X,T) into (Z,W). Then g(X) is closed in (Z,W) and $(g(X),W_{g(X)})$ is a closed image of (X,T) and, by Theorem 2.3, $(g(X), W_{g(X)})$ is ofoec.

(c) implies (a): If T is the indiscrete topology on X, then (X,T) is ofoec. Thus consider then case the T is not the indiscrete topology on X. Thus consider the case that T is not the indiscrete topology on X.

Since the identity function I from (X,T) onto itself is closed, then $(X,T)= (I(X),T_{I(X)})$ is ofoec.

A useful topological tool for studying properties of topological spaces has been, and continues to be, T_0-identification spaces. Below the relationships between ofoec spaces and T_0-identification spaces are resolved.

Definition 2.1. Let r be the equivalence relation on a space (X,T) defined by xry iff Cl({x})= Cl({y}). The T_0-identification space of (X,T) is $(X_0,Q(X_0))$, where X_0 is the set of r equivalence classes and $Q(X_0)$ is the decomposition topology on X_0 [10]. Let P be the natural map from (X,T) onto $(X_0,Q(X_0))$ and for each x in X, let C_x be the r equivalence class containing x.

Theorem 2.6. Let (X,T) be ofoec. Then $(X_0,Q(X_0))$ is ofoec.

Proof: If T is the indiscrete topology on X, then X_0 is a singleton set and $(X_0,Q(X_0))$ is ofoec. Thus consider the case that T is not the indiscrete topology on X. Since the natural map P from (X,T) onto $(X_0,Q(X_0))$ is open[6], then by Theorem 2.3, $(X_0,Q(X_0))$ is ofoec.

The following example shows the converse of Theorem 2.4 is not true.

Example 2.1. Let X= {a,b,c} and let T= { ϕ ,X,{a,b}}. Then (X,T) is not ofoec, but $(X_0,Q(X_0))$ is ofoec.

The discrete topology on a set with three or more elements can be used to easily show the continuous image of an ofoec space need not be ofoec. However, such is not the case for homeomorphisms.

Theorem 2.7. Onto functionally open equals closed is a topological property.

Proof: Let (X,T) be ofoec and let f be a homeomorphism from (X,T) onto a space (Y,S). If T is the indiscrete topology on X, then S is the indiscrete topology on Y and (Y,S) is ofoec. If T is not the indiscrete topology on X, then (Y,S) is an open image of the ofoec space (X,T) and by Theorem 2.3, (Y,S) is ofoec. Hence ofoec is a topological property.

The set X= {x,y} and T= { ϕ ,X,{x}} can be used to show the product of two ofoec spaces need not be ofoec.

3. Other Properties and Characterizations. Alexandroff spaces were so named[8] in 1982 in recognition of the initial work in the area done by P. Alexandroff.

Definition 3.1. A space (X,T) is Alexandroff iff T is closed under arbitrary intersections[8].

In 2010 Alexandroff [7] spaces were further investigated and characterized and strong Alexandroff spaces were introduced.

Theorem 3.1. Let (X,T) be a space and let C(T) denote the family of closed sets in (X,T). Then (X,T) is Alexandroff iff C(T) is a topology on X [7].

In the work below, C(T) will be used to denote the family of closed sets for a space (X,T).

Definition 3.2. A space (X,T) is strong Alexandroff iff T= C(T)[7].

Below Alexandroff and strong Alexandroff spaces are used to further investigate and characterize ofoec spaces.

Theorem 3.2. Let (X,T) be ofoec. Then (X,T) is Alexandroff.

Proof: If T= P(X) or T is the indiscrete topology on X, then (X,T) is both Alexandroff and strong Alexandroff. If X= {x,y} and T= $\{\phi, X, \{x\}\}$, then C(T)= $\{\phi, X, \{y\}\}$, which is a topology on X, and (X,T) is Alexandroff.

Theorem 3.3. Let (X,T) be ofoec. Then (X,T) and (X,C(T)) are homeomorphic.

Proof: If T= P(X) or T is the indiscrete topology on X, the identity function from (X,T) onto (X,C(T))= (X,T) is a homeomorphism. Thus consider the case that X= {x,y} and T= $\{\phi, X, \{x\}\}$. Then C(T)= $\{\phi, X, \{y\}\}$ and the function f from (X,T) onto (X,C(T)) defined by f(x)= y and f(y)= x is a homeomorphism.

Combining Theorem 2.7 with Theorem 3.3 gives the last result in this paper.

Corollary 3.1. Let (X,T) be a space. Then (X,T) is ofoec iff (X,C(T)) is ofoec.

Theorem 3.4. Let (X,T) be a space. Then (a) (X,T) is ofoec, strong Alexandroff iff (b) T = P(X) or T is the indiscrete topology on X.

Proof: (a) implies (b): Let C be closed in X and let $x \notin C$. Then C is open and $Cl(\{x\}) \subseteq X \backslash C$. Hence (X,T) is $R_{0,}$ ofoec and by Theorem 1.3, T= P(X) or T is the indiscrete topology on X.

Clearly, from the results above, (b) implies (a).

Thus the strong relationship between openness and closedness used in the definition of ofoec spaces is preserved when comparing the open sets in the space with the closed sets in the space.

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