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Connected domination in litact graph

M.H. MUDDEBIHAL and NAILA ANJUM

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Abstract

Let $G = (V, E)$ be a connected graph. The litact graph $m(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding members of G are adjacent or incident. A dominating set D of $m(G)$ is called a connected dominating set of $m(G)$ if the induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of D is called the connected domination number of $m(G)$ and is denoted by $\gamma_{mc}(G)$. In this paper, we initiate a study of this parameter. We obtain many bonds on $\gamma_{mc}(G)$ in terms of vertices, edges and different parameters of G and not the members of $m(G)$. Further we determine its relationship with other domination parameters.

Key words: Litact graph, domination number, connected domination number.

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Introduction

The graph $G=(V,E)$; we mean a finite,non trivial,undirected and connected graph with neither loops nor multiple edges. The number of vertices and edges of G are denoted by and respectively. For graph theoretic terminology we refer to[2].

In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v. The notations $\alpha_0(G)[\alpha_1(G)$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. Also $\beta_0(G)[\beta_1(G)]$ is the maximum number of vertices (edges) in a maximal independent set of vertex(edge) of G.

The maximum degree of a vertex v is denoted by $\Delta(G)$.

The concept of Litact graph of a graph

was introduced by *Kulli* and *Muddebihal*[3].

Let us now recall some standard definitions from domination theory.

Let $G=(V,E)$ be a graph. A set $D \subseteq V$ is said to be a dominating set, if $N[D]=V(G)$. The domination number of G, denoted by $\gamma'(G)$, is the minimum cardinality of a dominating set.

A set $S \subseteq E$ is an edge dominating set, if every edge not in S is adjacent to atleast one edge in S. The edge domination number of G, denoted by $\gamma'(G)$, is the minimum cardinality of an edge dominating set.

A set D is a total dominating set if for every vertex $v \in V$, there exists a vertex $u \in D$, $u \neq v$ such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set.

A connected dominating set D is a dominating set whose induced subgraph $\langle D \rangle$ is connected. The connected domination number of a graph G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set. Similarly the connected edge domination number $\gamma_c'(G)$ is the minimum cardinality of a connected edge dominating set.

The domination number of $m(G)$ is denoted by $\gamma_m(G)$ and is the minimum cardinality of a dominating set of $m(G)$.

We, define the connected domination number in litact graph as follows.

The dominating set D of $m(G)$ is called connected dominating set of $m(G)$ if the induced sub graph $\langle D \rangle$ is connected. The connected domination number equals the minimum cardinality of D and is denoted by $\gamma_{mc}(G)$.

In this paper, we obtain certain bounds on $\gamma_{mc}(G)$ in terms of vertices, edges and other parameters of G. Also we determine its relationship with other domination parameters of G.

We need the following Theorems to prove our later results.

Theorem[1] *[A]* : For any connected (p,q) graph $\gamma'(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$,

Theorem[4] *[B]* : For any tree T of order p with at leasttwo cut vertices $\gamma_c'(T) = p - 1 - n$ where n is the number of end vertices in T.

Theorem[4] *[C]*: For any graph G, $\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \gamma(G)$.

2. Results

First we list out the exact values of $\gamma_{mc}(G)$ for some standard graphs.

Theorem 1:

a. For any path P_p, with $p \geq 4$ vertices

$$\gamma_{mc}(P_p) = p - 3$$

b. For any cycle C_p,

$$\gamma_{mc}(C_p) = p - 2$$

c. For any star $K_{1,p}$,

$$\gamma_{mc}(K_{1,p}) = 1$$

d. For any wheel W_p,

$$\gamma_{mc}(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$$

In the following Theorems, we obtain the lower bound for $\gamma_{mc}(G)$.

Theorem 2: For any non-trivial connected graph G, $\gamma(G) \leq \gamma_{mc}(G) + 1$.

Proof: Let D be a minimal dominating set in G. Now, consider $S=\{e_1, e_2, \ldots \ldots \ldots, e_i\}$ be the minimal edge dominating set in G and $C=\{c_1, c_2, \ldots \ldots \ldots, c_n\}$ be thesetof cut verticesin G. Then the minimal set of vertices $D_1 \in V[\, m(G)]$ such that $D_1 \subseteq S \cup C$ and $N[D_1] = V[\, m(G)]$ is the minimal dominating set in $m(G)$. Further, if $D_2 \subseteq V[\, m(G)] - D_1$ and $D_2 \in N(D_1)$, then we consider aset $D_2' \subset D_2$ suchthat $D_2' \cup D_1$ forms a minimal connected dominating set in $m(G)$. Then clearly $|D| \subseteq |D_2' \cup D_1| + 1$ which follows $\gamma(G) \leq \gamma_{mc}(G) + 1$.

Theorem 3: For any connected graph G , $\gamma_m(G) \leq \gamma_{mc}(G)$.

Proof: Let $D_1 = \{v_i \backslash v_i \in V[m(G)]\}$ be the minimal dominating set in $m(G)$. Also let $D_2 \subseteq V[\, m(G)] - D_1$ and $D_2 \in N(v_i)$ $\forall\, v_i \in D_1$. Further, we consider a sets $D_2' \subset D_2$ such that $D_2' \cup D_1$ forms a minimal connected dominating set in $m(G)$. Now, since every connected dominating set is a dominating set, we have $|D_1| \subseteq |D_2' \cup D_1|$ which gives $\gamma_m(G) \leq \gamma_{mc}(G)$.

Theorem 4: For any connected (p,q) graph, $\left\lfloor \frac{p}{\Delta(G)} \right\rfloor \leq \gamma_{mc}(G)$ where

$$G \ncong P_2, P_4 .$$

Proof : Suppose $D_1 = \{v_1, v_2, \ldots \ldots \ldots, v_i\}$ $\forall\, v_k, 1 \leq k \leq i$ be the minimal set of vertices in $m(G)$ such that $N[D1] = V[\, m(G)]$. Also let $D_2 = \{v_1', v_2', \ldots \ldots \ldots, v_i'\}$ $\forall\, v_k', 1 \leq k \leq i$, be a set of vertices in $m(G)$ and $D_2 \in N(v_i)$ $\forall\, v_i \in D_1$ suchthat $v_i \cup v_i'$ forms exactly one minimal connected path between every pairof vertices of D_2. Then clearly $D_1 \cup D_2$ is the minimal connected dominating set in $m(G)$. Further, let $F = \{u_1, u_2, \ldots \ldots \ldots, u_i\}$ $\forall\, u_k, 1 \leq k \leq i$ be the set of all non-end vertices in G, then there exists at least one vertex $u_k \in F$ of maximum degree $\Delta(G)$, such that $|D_1 \cup D_2| \geq \left\lfloor \frac{p}{\Delta(G)} \right\rfloor$. Hence the result follows.

Now if $G \cong P_2, P_4$ then $|D_1 \cup D_2| = 1$, and the fraction $\left\lfloor \frac{p}{\Delta(G)} \right\rfloor = 2$. This contradicts the result.

Theorem 5: For any (p,q) graph G with maximum degree $\Delta(G)$

$$\left\lceil \frac{p}{(1+\Delta(G))} \right\rceil - 1 \leq \gamma_{mc}(G).$$

Proof: Let $G(p,q)$ be any connected graph with maximum degree $\Delta(G)$. Then by *Theorem [C]* we have $\left\lceil \frac{p}{(1+\Delta(G))} \right\rceil \leq \gamma(G)$. And using *Theorem1*, we get $\left\lceil \frac{p}{(1+\Delta(G))} \right\rceil \leq \gamma_{mc}(G)+1$ which gives the result.

Theorem 6: If G is a connected graph, then $\left\lfloor \frac{diam(G)+1}{2} \right\rfloor \leq \gamma_{mc}(G)$ provided that G is not a tree with $diam(T) = 3$.

Proof: Let $S = \{e_1, e_2, \dots \dots \dots, e_j\}$ be the set of edges in G which constitute the diametral path in G. Clearly $|S| = diam(G)$. Now, without loss of generality, let D_1 be a minimal dominating set in $m(G)$ and $D_2 \subseteq V[\,m(G)] - D_1$ such that $D_2 \in N(D_1)$. Then if $D_2' \subset D_2$ such that $\langle D_2' \cup D_1 \rangle$ is the minimal connected sub graph of $m(G)$ then $D_2' \cup D_1$ gives the minimal dominating set in $m(G)$. Further since $S \subseteq V[\,m(G)]$ and $D_2' \cup D_1$ is a γ_{mc} set, the diametral path includes atmost $\gamma_{mc}(G) - 1$ edges joining the neighborhoods of the vertices of $D_2' \cup D_1$. Hence $diam(G) \leq \gamma_{mc}(G) + \gamma_{mc}(G) - 1$ which gives $\left\lfloor \frac{diam(G)+1}{2} \right\rfloor \leq \gamma_{mc}(G)$.

Now, suppose G be a tree T with $diam(T) = 3$. let e_1, e_2, and e_3 be the edges which constitute the diametral path in T. Clearly e_2 is the only non-end edge in T such that $N[e_2] = E(T)$. Thus the corresponding vertex $e_2 = v \in V[\,m(T)]$ is such that $N[v] = V[\,m(T)]$. Hence $|D_2' \cup D_1| = 1$. Clearly $diam(G) > \gamma_{mc}(G) + \gamma_{mc}(G) - 1$ which gives $\left\lfloor \frac{diam(G)+1}{2} \right\rfloor > \gamma_{mc}(G)$. which is a contradiction.

The next two Theorems gives the upper bound for $\gamma_{mc}(G)$ in terms of the vertices of G.

Theorem 7: For any (p,q) connected graph G, $\gamma_{mc}(G) \leq \left\lceil \frac{p}{2} \right\rceil + 2$.

Proof: Let $S = \{e_1, e_2, \dots \dots \dots, e_n\}$ be the minimal edge dominating set in G and let $C = \{c_1, c_2, \dots \dots \dots, c_n\}$ be the set of cutvertices in G. Then $S \cup C \subseteq V[\,m(G)]$. Now, we consider a minimal set of vertices $D_1 \subseteq S \cup C$ in $m(G)$ such that $N[D_1] = V[m(G)]$. Then D_1 is the minimal dominating set in $m(G)$. Further let $D_2 \subseteq V[m(G)] - D_1$ and $D_2 \in N(D_1)$. Now if $D_2' \subset D_2$ such that $\langle D_1 \cup D_2' \rangle$ is the minimal connected sub graph of $m(G)$ then $D_1 \cup D_2'$ is the minimal connected dominating set in $m(G)$. since $S \subseteq V[m(G)]$ and also by *Theorem [A]* we get $|D_1 \cup D_2'| \leq \left\lceil \frac{p}{2} \right\rceil + 2$. Therefore $\gamma_{mc}(G) \leq \left\lceil \frac{p}{2} \right\rceil + 2$.

Theorem 8: For any connected non-trivial graph G, $\gamma_{mc}(G) \leq p - m$, where m is the number of end vertices in G. Further the equality holds for $G \cong K_{1,p}$.

Proof : For $diam(G) \leq 2$ the result is obvious. Hence we consider $diam(G) \geq 3$. Now, let $S = \{v_1, v_2, \dots \dots \dots, v_n\}$ be the set of end vertices of G with $|S| = m$. Now let $F \subseteq E(G)$ be an edge dominating set in G and $C = \{c_1, c_2, \dots \dots \dots, c_n\}$ be the set of cut vertices in G. Then the minimal set of vertices $D_1 \in V[\, m(G)]$ where $D_1 \subseteq S \cup C$ such that $N[D_1] = V[\, m(G)]$ formsa minimal dominating set in $m(G)$. Here, we consider the following cases.

Case 1: Suppose $m = \phi$ and we consider $D_2 = \{v_1, v_2, \dots \dots \dots, v_j\}$ such that $\forall\, v_j \in N(D_1)$; $1 \leq i \leq j$. Then $F = \{v_1, v_2, \dots \dots \dots, v_i\}$ is a finite set and $F \subset D_2$. Clearly $\langle F \cup D_1 \rangle$ is connected and $F \cup D_1$ is minimal connected dominating set in $m(G)$. Hence $|F \cup D_1| \leq V(G) - |S|$ which gives $\gamma_{mc}(G) \leq p - m$

Case 2: Further if $m = \phi$ then there exists at least one end vertex v_i, $1 \leq i \leq p$, again we consider the finite set F and the set D_1, which gives $\langle F \cup D_1 \rangle$ which is connected. Hence $|F \cup D_1| \leq V(G) - |S|$ which gives $\gamma_{mc}(G) \leq p - m$.

Suppose $G \cong k_{1,p}$, then we have $\gamma_{mc}(K_{1,p}) = 1$ and $p=m+1$. Hence equality follows.

In the following Theorem, we relate $\gamma_{mc}(T)$ and $\gamma'_c(T)$.

Theorem 9: For any tree T; $\gamma_{mc}(T) = \gamma'_c(T)$.

Proof: Suppose G is a Tree T. Let $I = \{v_1, v_2, \dots \dots \dots, v_n\}$ be the set of all end vertices of T. Then the set $F=\{e_1, e_2, \dots \dots \dots, e_j\}$ of all the edges which are not incident to the vertices of I, forms a connected edge dominating set of T. Now without loss of generality the corresponding set F generates the set of vertices which forms a connected dominating set in $m(G)$. Hence clearly $|D| = |F|$ which gives $\gamma_{mc}(T) = \gamma'_c(T)$.

Theorem 10: For any non-trivial tree T with S non-end edges $\gamma_{mc}(T) = S$ where $S \neq \phi$

Proof: Let S be the set of non-end edges in G. By *Theorem 9* we have $\gamma_{mc}(T) = \gamma'_c(T)$. Now by *Theorem [B]*, we know $\gamma'_c(T) = p - 1 - n$. Since for a tree $q=p$-1, then $\gamma'_c(T) = q - n$ which gives $\gamma'_c(T) = S$. Finally we get $\gamma_{mc}(T) = S$.

The following Theorem relates $\gamma_{mc}(G)$ and $\alpha_1(G)$

Theorem 11: For any connected graph G, $\gamma_{mc}(G) < 2\alpha_1(G)$.

Proof : Suppose $S=\{e_1, e_2, \dots \dots \dots, e_j\}$ be the set of all end edges in G. Then $S \cup J$ where $J \subseteq E(G) - S$ be the minimal set of edges which covers all the vertices of G such that $|S \cup J| = \alpha_1(G)$. Further let $D_1=\{u_i \backslash u_i \in V[m(G)]\}$ be the minimal dominating set in $m(G)$. Now if $D_2 = \{u_i' \backslash u_i' \in N(u_i)\ \forall\ u_i \in D_1\}$ such that $u_i' \cup u_i$ forms exactly a unique minimal connected path between every pair of vertices of D_2; then clearly $D_1 \cup D_2$ is the minimal connected dominating set in $m(G)$. Since $V[m(G)]=E(G) \cup C(G)$ where $C(G)$ is the set of all cut vertices in G. But $E(G) \subseteq V[m(G)]$ such that $S \subset V[m(G)]$ which gives contribution to the set $E(G)$ to represent twice to the edge covering number in G. Hence $|D_1 \cup D_2| < 2|S \cup J|$ which gives $\gamma_{mc}(G) < 2\alpha_1(G)$.

The next theorem relates $\gamma_{mc}(G)$ and $\gamma_t(G)$ in terms of vertex covering number of G.

Theorem 12: For any (p,q) graph G, $\gamma_{mc}(G) + \gamma_t(G) > \alpha_0(G)$.

Proof : Let $B=\{v_1, v_2, \dots \dots \dots, v_i\}$ be a set of vertices in G such that $|B|=\alpha_0(G)$. Further, let D be the minimal dominating set in G. Suppose $V_1=V(G)-D$ and $H \subseteq V_1$ such that $H \in N(D)$ in G. Then $D \cup H$ is a total dominating set in G. Now, consider a set $F \subseteq E(G)$ be the edge dominating set in G and $C \in V(G)$ be the set of cut vertices in G. Then the set $D_1 \subseteq F \cup C$ in $m(G)$ such that $N[D_1]=V[m(G)]$ is the minimal dominating set in $m(G)$. Again let $D_2 \subseteq V[m(G)]-D_1$ and $D_2 \in N(D_1)$. Then if $D_2' \subset D_2$ such that $\langle D_1 \cup D_2' \rangle$ forms a minimal connected sub graph of $m(G)$, then $D_1 \cup D_2'$ is the minimal connected dominating set in $m(G)$. Hence clearly $|D_1 \cup D_2'| \cup |D \cup H| > |B|$ which gives $\gamma_{mc}(G) + \gamma_t(G) > \alpha_0(G)$.

Theorem 13: For any connected graph, $\gamma_{mc}(G) \leq \gamma(G) + \gamma_t(G)$.

Proof: Let D be the minimal set of vertices in G such that $N[D]=V(G)$ and $V'=V(G)-D$. Also let $H \subseteq V'$ and $H \in N(D)$. Then $\langle H \cup D \rangle$ such that $\deg u_i \neq 0, \forall\ u_i \in D \cup H$ forms a total dominating set in G. Now, let $S \in E(G)$ be the minimal edge dominating set in G and $C \in V(G)$ be the set of cut vertices. Then $D_1 \in V[m(G)]$ such that $D_1 \subseteq S \cup C$ such that $N[D_1]=V[m(G)]$ is the minimal dominating set in $m(G)$. Further let $D_2 \subseteq V[m(G)]-D_1$ and $D_2 \in N(D_1)$. Then consider a set $D_2' \subset D_2$ such that $D_1 \cup D_2'$ is the minimal connected dominating set in $m(G)$. Hence clearly $|D_1 \cup D_2'| \subseteq |D| \cup |D \cup H|$ which gives $\gamma_{mc}(G) \leq \gamma(G) + \gamma_t(G)$.

Theorem 14: For any (p,q) graph G, $\gamma_{mc}(G) + \gamma(G) \leq p + \left\lceil \frac{\alpha_0}{2} \right\rceil$.

Proof : Let $B=\{v_1, v_2, \dots \dots \dots, v_i\}$ be the minimum number of vertices which covers all edges of G such that $|B|=\alpha_0$. Further let D be the γ *set* of G. Suppose $F=\{e_1, e_2, \dots \dots \dots, e_k\}$ be the set of edges in G which forms a minimal connected edge dominating set in G. Let $C \in V(G)$ be the set of cut vertices in G. Then the set $D_1 \in V[m(G)]$ where $D_1 \subseteq F \cup C$ such that $N[D_1]=V[m(G)]$ is the minimal dominating set in $m(G)$. Further let $D_2 \subseteq V[m(G)]-D_1$ and $D_2 \in N(D_1)$ then if we consider a set $D_2' \subset D_2$ such that $\langle D_1 \cup D_2' \rangle$ is the minimal connected sub graph of $m(G)$ then $D_1 \cup D_2'$ is the minimal connected dominating set of $m(G)$ such that $|D_1 \cup D_2| \cup |D| \leq p + \left\lceil \frac{\alpha_0}{2} \right\rceil$. Hence $\gamma_{mc}(G)$ $+ \gamma(G) \leq p + \left\lceil \frac{\alpha_0}{2} \right\rceil$.

The next theorem gives the Nordhaus_Gaddumtype of result

Theorem 15 :

I. $\gamma_{mc}(G) \,.\, \gamma_{mc}(\overline{G}) \leq p + q$

II. $\gamma_{mc}(G) + \gamma_{mc}(\overline{G}) \leq p$

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