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Some Universal Objects of Fuzzy Topology

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Abstract

In this paper fuzzy setting concepts pf some universal objects like fuzzy products, fuzzy equalising objects, fuzzy retreat objects and their duals have been through continuous mappings for fuzzy topological spaces, Their various properties are also studied.

Key words : Fuzzy topological space, Fuzzy product, Equalising, fuzzy object, fuzzy retreat and duals.

1. Introduction

While studying the fuzzy setting generalisations of many topological aspects of fuzzy topology by chang[1], we come across some universal objects such as product, equalizer, retreat and their duals whose fuzzy setting are yet to be done. In this paper, we introduce those fuzzy setting concepts and discuss their properties :

As a priliminary out look, let U be a non empty set. A fuzzy λ in U is a mapping $\lambda : U \rightarrow [0,1]u$ and O_u being the constant fuzzy sets with values 1 and 0 respectively.

Let $IU = \{\lambda, \mu, \nu \ldots\ldots\}$ denote the family of all fuzzy sets in U and (U, J), a fuzzy topological space. A fuzzy fucntion through fuzzy point u is defined in [0, 1] by the usual concept of Zadeh[2] that given any function $f : U \rightarrow V$ and any fuzzy set λ in U with $f\lambda(v) = V\{1(u) 1 f(u) = v\}$, we always have f(ui) = f(u)i.

Fuzzy product object and its dual :

Let us first generalise the concept of fuzzy product appeared in the work of Hutton[3] by the following proposition.

Propositions (2.1) :

Let $\{\lambda\alpha : \alpha \in L = [0,1]\}$ be a class of fuzzy topological spaces, Then (a) An ordered pair $(J, \tau\alpha)$ of any fuzzy topological space with a fuzzy continuous mapping $\alpha : J \rightarrow \lambda\alpha$

is available for each $\alpha \in L$

(b) Corresponding to fuzzy topological space (U,J) and a chain $\{r_\alpha : U \to \lambda\alpha\}$ of fuzzy continuous mappings, we have a unique fuzzy continuous mapping $s : U \to J$ such that $t_\alpha s = r_\alpha$ for each $\alpha \in L$

Proof :

(a) Let $J = \lambda\alpha_1 \times \lambda\alpha_2 \times \lambda\alpha_3 \times \ldots\ldots..$ be the cartesian product aloing with the fuzzy product topology of Wong[4]. This fuzzy product topology is the weakest one that makes the projections.

$t\alpha : \lambda\alpha_1 \times \lambda\alpha_2 \times \lambda\alpha_3 \times \ldots\ldots\ldots\ldots \to \lambda\alpha$ of fuzzy continuous mappings, thus forming a class of fuzzy open sets $t\alpha^{-1}(X\alpha)$ which is evidently a sub-base for the fuzzy product topology on $\lambda\alpha_1 \times \lambda\alpha_2 \times \lambda\alpha_3 \times \ldots\ldots.$; $X\alpha$ being fuzzy in $\lambda\alpha$.

(b) For all fuzzy points $u_1 \in U$, define $s : U \to T$ by $s(u_i) = \{r\alpha(u_i)\}\alpha \in L$. Then $t\alpha s = r\alpha$ and s is fuzzy continuous because for any open set $Y = t\alpha^{-1}(X\alpha)$ belonging to the sub-base of J, $s^{-1}(Y)$ $(X\alpha) = r\alpha$-1 $(X\alpha)$ is fuzzy open in U for each $\alpha \in L$.

Definition (2.2) :

The ordered pair (J, $t\alpha$) is called the fuzzy product of the class $\{\lambda\alpha : \alpha \in L\}$ of fuzzy topological spaces, the cancellation law for which is indicated as follows :

Propositions (2.3) :

If $(\Im, t\alpha)$ be a fuzzy product of the class $\{\lambda\varepsilon : \alpha \in L\}$ of fuzzy topological spaces and $r, k : U \to \tau$

(U being a fuzzy topological space) are fuzzy continuous mappings, then $t\alpha\gamma = t\alpha K(\alpha \in L)$ implies r = k

Proof : In considering the class $\{t\alpha r : U \to \lambda\alpha, \alpha \in L\}$ of fuzzy continuous mappings, it follows from the second part of proposition (2.1).

Proposition (2.4) :

The two fuzzy products of the class are fuzzy homeopophic

Proof : Let $(\Im, t\alpha)$ and $(S, s\alpha)$ and two fuzzy products of the class $\{\lambda\alpha : \alpha \in L\}$ of fuzzy topological spaces, then by second part of proposition (2.1) a unique fuzzy continuous mapping $r : s \to T$ exists such that $t\alpha r = s\alpha$ a for each $\alpha \in L$ and unique fuzzy continuous mapping $K : T \to S$ for each $\alpha \in L$.

Thus $t\alpha = s\alpha K = t\alpha rk$ for each $\alpha \in L$.

Again as $t\alpha = t\alpha 1\tau$ for each $\alpha \in L$, it follows from the factorization condition of fuzzy products that $rK = 1\tau$. Similarly it can be proved that Kr=1s. So τ and S are fuzzy homeoporphic.

As regards the duality of fuzzy product, we generalise the concept of Wong[4,5] as follows:

Proposition (2.5) :

For a class $\{\lambda\alpha : \alpha \in L\}$ of fuzzy topological spaces,

(a) An ordered pair $(\Im' t\alpha')$ of any fuzzy topological space with a fuzzy continuous mapping $t\alpha' : J' \to \lambda\alpha$ is available for each $\alpha \in L$.

(b) Corresponding to fuzzy topological space U and a chain $\{r\alpha : U \to \lambda\alpha, \alpha \in L\}$ of fuzzy continuous mapping, we have a unique fuzzy continuous mapping $s : J' \to U$ such that

$t\alpha' s = r\alpha$ for each $\alpha \in L$

Proof :

(a) Let J' be the union of disjoint fuzzy topological spaces $\lambda\alpha \times \{\alpha\}$ i.e.

$J' + U(\lambda\alpha \times \{\alpha\})$.

A fuzzy set X is fuzzy open in J' if $X \cap (\lambda\alpha \times \{\alpha\}$ is fuzzy open in $\lambda\alpha \times \{\alpha\}$ for each $\alpha \in L$.

We observe that the fuzzy inclusion maps $t\alpha' : \lambda\alpha \to J'$ by $t\alpha'(zi, \alpha), zi \in \lambda\alpha$ is continuous.

(b) Define $s : J' \to U by\, s(zi, \alpha) = r\alpha(zi)$ clearly $st\alpha' = r\alpha$. To prove that s is fuzzy continuous, consider any fuzzy open set Y in U.

Then $\underset{\alpha\ L}{U}\left(s^{-1}(\gamma) \cap \lambda\alpha \times \{\alpha\}\right) = \underset{\alpha \in L}{U} r^{-1}\alpha(\gamma)$

Definition (2.6) : The ordered pair $(J', t\alpha')$ is called the dual of the fuzzy product of the class $\{\lambda\alpha : \alpha \in L\}$ fuzzy topological space, the cancellation law for which is indicated as follows :

Proposition (2.7) : Let $(J', t\alpha')$ be a dual of fuzzy product of the class $\{\lambda\alpha : \varepsilon \in L\}$ of fuzzy topological space and $r, K : U \to J'$ being a fuzzy topological sace) are fuzzy continuous mappings, then $rt\alpha' = Kt\alpha' \to r = K$.

Proof :

The proof follows by proposition (2.3)

Proposition (2.8) :

The two duals of the fuzzy product are fuzzy homeomorphic.

Proof :

Followed by proposition (2.4)

Fuzzy equalising object and its dual :

We shall now deal with another universal object namely the equalising fuzzy object.

Theorem (3.1) :

Let $f, g : \lambda \to$ be a pair of fuzzy continuous mappings. Then

(a) There exists a fuzzy topological space σ and a fuzzy continuous mapping h : $\sigma \to \lambda$ such that fh = gh.

(b) For any fuzzy topological space U and fuzzy continuous mapping r : U $\to \lambda$ satisfying fr = gr, there exists a unique fuzzy continuous mapping $s : U \to \sigma$ such that r = hs

Proof :

(a) Let $\sigma = \{u, \lambda : f(u_i) = g(u_i)\}$ we impose the sub space topological of Ghanim[6] on σ. The inclusion map $h : \sigma \to U$ is defined by $h(u_i) = u_i$, $u_i\ \sigma$. Clearly h is fuzzy continuous[4] Wong and fh = gh.

(b) Let $s : U \to \sigma$ be defined by $s(ui) = r(ui)$, ui U. Since $f(r(ui)) = fr(ui) = gr(ui)$

$= g(r(ui))$, we see that (ui) σ

Again as $hs(ui) = h(s(ui)) = s(ui) = r(ui)$

$ui \in U$, we have hs=r. s is fuzzy continuous because for any fuzzy open set X in σ. $s^{-1}(X) = s^{-1}(h^{-1}(X)) = r^{-1}(X)$ in fuzzy open in U due to the fact that r is fuzzy continuous. Clearly s is unique.

Definition (3.2) :

The ordered pair (σ, h) in theorem (3.1) is called fuzzy equalsin object of the fuzzy continuous mapping $f, g : \lambda \to \mu$.

Theorem (3.3) :

Let $f, g : \lambda \to \mu$ be a pair of fuzzy continuouos mappings. Then

(a) There exists a fuzzy topological space ρ and fuzzy continuous mapping $e : \mu \to \rho$ such that ef = eg.

(b) For any fuzzy topological space U and fuzzy continuous mapping $r : m \to U$ satisfying rf = rg there exists a unique fuzzy continuous mapping $s : \rho \to U$ such that r = se.

Proof :

(a) Let v be the smallet fuzzy subset of $\mu \times \mu$ containing[5]

$$\tilde{\tilde{\rho}}\{f(ui), g(ui) | ui \in \lambda\} \subseteq \mu \times \mu$$

Which defines a fuzzy equivalance relation $\sim$ on μ.

For vj m, let $[[wj]]$ denote the fuzzy equivalence class of wj. If $\rho = \mu / \sim$, r has the quotient fuzzy topology (Wong [4]).

Agains for any uk 1, f(uk) ~ g(uk), it follows that

$$e(f(uk)) = [[f(uk)]] = [[g(uk)]] = e(g(uk))$$
$$\to ef = eg$$

(b) Let $\sigma : \rho \to U$ be a fuzzy mapping defined by

$$s[[wj]] = r(wj) \quad .$$

Let $[[wj]] = [[zl]]\rho$, then e(wj) = e(zl) where (wj, zl) ν

If we define $\nu_e = \{(wj, zi)\ m \times m | r(wj)$ $= z r(zl)\}$

Then it is easy to see that ne is a fuzzy equivalence relation.

Again semi $r(f(uk)) = (g(uk))$ for all uk λ, it follows that $(f(uk), g(uk))\ \nu_e$. Hence ν_e.

But ν is the smallest fuzzy equivalance relation on m containing ρ, so $\nu_e\ \nu$.

Thus (wj, zl) $\nu\ \ \nu_e$ and r(wj) = r(zl)

Conseqiently s is well defined, unique and fuzzy continuous and r = se.

Definition (3.4) :

The ordered pair (p, e) in theorem (3.3) is called dual of the fuzzy equalising object τ fuzzy continuous mapping $f, g : \lambda \to \mu$

4. Fuzzy Retreat Object and Its Dual :

Let us concentrate overselves to the third universal object which is fuzzy retreat object.

Theorem 4.1 :

For a pair of fuzzy continuous mappings $f : \lambda \to \xi, g : u \to \xi,$ the following hold.

(a) A fuzzy topological space G and two fuzzy continuous mappings $\eta\ G : \to \lambda$ can be found such that $f\eta = g\delta$.

(b) For any fuzzy topological space U and fuzzy continuous mapping $r : U \to l,$ $K : \to m$ satisfying fr = gk, then exists a unique fuzzy continuous mapping $s : U \to G$ such that

$$r = nD, k = \delta D$$

Proof :

(a) Let $G = \{(uk, wj) l \times m | (uk) g(wj)\} \subseteq \lambda \times \mu$

There l x m has product fuzzy topology[4] and G has a fuzzy sub space topology (Ghanim [6]).

(b) Obvious

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