Ultra Scientist Vol. 24(3)A, 535-542 (2012).

A fixed point theorem on n metric spaces

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(Acceptance Date 1st December, 2012)

Abstract

In this paper a fixed point theorem on n metric spaces is proved. This theorem generalizes and extends the results obtained[1-3] in from three and four metric spaces to n metric spaces.

Keywords: fixed point, metric space, complete metric space.

Mathematical Subject Classification: 47H10, 54H25.

1. Introduction

Jain *et al.*[1] introduced a related fixed point theorem on three metric spaces. Luljeta and Kristaq[3] generalized it on three metric spaces to four metric spaces. In this paper we shall establish a more generalized fixed point theorem on n metric spaces.

Throughout this paper the product $T_n T_{n-1} T_{n-2} \ldots\ldots\ldots\ldots T_1$ is denoted by $\prod_1^n T_i$.

2. Main results

We will give and prove our theorem as follows:

Theorem 2.1: Let (X_i, d_i), $i = 1, 2, 3 \ldots\ldots n$, and $T_i : X_i \to X_{i+1}$, $i = 1, 2, 3 \ldots\ldots n-1$ with $T_n : X_n \to X_1$ be the n mappings satisfying the following inequalities:

$$d_1\left(\prod_2^n T_i\, x_2, \prod_1^n T_i\, x_1\right) \le c\, \frac{f_1(x_1, x_2)}{g_1(x_1, x_2)} \quad (1.1)$$

$$d_2\left(\prod_3^1 T_i x_3, \prod_2^1 T_i\, x_2\right) \le c\, \frac{f_2(x_2, x_3)}{g_2(x_2, x_3)} \quad (1.2)$$

$$d_3\left(\prod_4^2 T_i x_4, \prod_3^2 T_i\, x_3\right) \le c\, \frac{f_3(x_3, x_4)}{g_3(x_3, x_4)} \quad (1.3)$$

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$$d_n\left(\prod_1^{n-1} T_i x_1, \prod_{n-1}^n T_i\, x_n\right) \le c\, \frac{f_n(x_n, x_1)}{g_n(x_n, x_1)} \quad (1.n)$$

for all $x_i \in X_i$, $i = 1, 2, 3 \ldots\ldots\ldots n$ and for which $g_i(x_i, x_{i+1}) \neq 0$, $i = 1, 2, 3 \ldots\ldots\ldots n-1$ with $g_n(x_n, x_1) \neq 0$, where $0 \le c < 1$ and

$$f_1(x_1, x_2)=\max.d_1(x_1, \prod_{1}^{n} T_i\, x_1)d_n(\prod_{2}^{n-1} T_i\, x_2, \prod_{1}^{n-1} T_i\, x_1);\ d_1(x_1, \prod_{1}^{n} T_i\, x_1)d_{n-1}(\prod_{2}^{n-2} T_i\, x_2, \prod_{1}^{n-2} T_i\, x_1);$$

$$d_1(x_1, \prod_{1}^{n} T_i\, x_1)d_{n-2}(\prod_{2}^{n-3} T_i\, x_2, \prod_{1}^{n-3} T_i\, x_1)\ldots\ldots\ldots\ldots$$

$$d_1(x_1, \prod_{1}^{n} T_i\, x_1)d_3(T_2x_2, T_2T_1x_1);$$

$$d_1(x_1, \prod_{1}^{n} T_i\, x_1)d_2(x_2, \prod_{2}^{1} T_i\, x_2);\ d_1(x_1, \prod_{2}^{n} T_i\, x_2)\, d_2(x_2, T_1x_1)\}.$$

$$f_2(x_2, x_3) = \max.\ \{d_2(x_2, \prod_{2}^{1} T_i\, x_2)d_1(\prod_{3}^{n} T_i\, x_3, \prod_{2}^{n} T_i\, x_2);\ d_2(x_2, \prod_{2}^{1} T_i\, x_2)d_n(\prod_{3}^{n-3} T_i x_2, \prod_{2}^{n-2} T_i\, x_2);$$

$$d_2(x_2, \prod_{2}^{1} T_i\, x_2)d_{n-1}(\prod_{3}^{n-2} T_i\, x_3, \prod_{2}^{n-3} T_i\, x_2)\ldots\ldots\ldots\ldots$$

$$\ldots\ldots d_2(x_2, \prod_{2}^{1} T_i\, x_2)d_4(T_3x_3, T_3T_2x_2);$$

$$d_2(x_2, \prod_{2}^{1} T_i\, x_2)d_3(x_3, \prod_{3}^{2} T_i\, x_3);\ d_2(x_2, \prod_{3}^{1} T_i\, x_3)\, d_3(x_3, T_2x_2)\}.$$

$$f_3(x_3, x_4) = \max.\ \{d_3(x_3, \prod_{3}^{2} T_i\, x_3)d_2(\prod_{4}^{1} T_i\, x_4, \prod_{3}^{1} T_i\, x_3);\ d_3(x_3, \prod_{3}^{2} T_i x_3)d_1(\prod_{4}^{n} T_i\, x_4, \prod_{3}^{n} T_i\, x_3);$$

$$d_3(x_3, \prod_{3}^{2} T_i\, x_3)d_n(\prod_{4}^{n-1} T_i\, x_4, \prod_{3}^{n-1} T_i\, x_3)\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots d_3(x_3, \prod_{3}^{2} T_i\, x_3)d_5(T_4x_4, T_4T_3x_3);$$

$$d_3(x_3, \prod_{3}^{2} T_i\, x_3)d_4(x_4, \prod_{4}^{3} T_i\, x_4);\ d_3(x_3, \prod_{4}^{2} T_i\, x_4)\, d_4(x_4, T_3x_3)\}.$$

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$$f_n(x_n, x_1)=\max.\{d_n(x_n, \prod_{n}^{n-1} T_i\, x_n)d_{n-1}(\prod_{1}^{n-2} T_i\, x_1, \prod_{n}^{n-2} T_i\, x_n);\ d_n(x_n, \prod_{n}^{n-1} T_i\, x_n)d_{n-2}(\prod_{1}^{n-3} T_i\, x_1,$$

$$\prod_{n}^{n-3} T_i\, x_n);\ d_n(x_n, \prod_{n}^{n-1} T_i\, x_n)d_{n-3}(\prod_{1}^{n-4} T_i\, x_1, \prod_{1}^{n-4} T_i\, x_1)\ldots\ldots\ldots\ldots\ldots\ldots d_n(x_n, \prod_{n}^{n-1} T_i\, x_n)$$

$$d_2(T_1x_1, T_1T_nx_n);\ d_n(x_n, \prod_{n}^{n-1} T_i\, x_n)d_1(x_1, \prod_{1}^{n} T_i\, x_1);\ d_n(x_n, \prod_{n}^{n-1} T_i\, x_1)d_2(x_1, T_nx_n)\}.$$

$$g_1(x_1, x_2)=\max.\{d_1(x_1, \prod_{2}^{n} T_i\, x_2);\ d_1(x_1, \prod_{1}^{n} T_i x_1);\ d_2\,(T_1x_1, \prod_{2}^{1} T_i\, x_2)\}.$$

$$g_2(x_2, x_3)=\max.\{d_2(x_2, \prod_{3}^{1} T_i x_3);\ d_2(x_2, \prod_{2}^{1} T_i x_2);$$

$d_3(T_2x_2, \prod_3^2 T_i\ x_3)\}$.

$g_3(x_3, x_4) = \max.\{d_3(x_3, \prod_4^2 T_i\ x_4); d_3(x_3, \prod_3^2 T_i$

$x_3); d_4(T_3x_3, \prod_4^3 T_i\ x_4)\}$.

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$g_n(x_n, x_1)=\max.\{d_n(x_n, \prod_1^{n-1} T_i x_1); d_n(x_n, \prod_n^{n-1} T_i\ x_n);$

$d_1(T_nx_n, \prod_1^n T_i\ x_1)\}$.

then $\prod_1^n T_i$ has a unique fixed point $\alpha_1 \in X_1$, $\prod_2^1 T_i$ has a unique fixed point $\alpha_2 \in X_2$.......... $\prod_n^{n-1} T_i$ has a unique fixed $\alpha_n \in X_n$. Further $T_1\alpha_1 = \alpha_2$, $T_2\alpha_2 = \alpha_3$............ $T_n\alpha_n = \alpha_1$.

Proof: Let $x_1 \in X$ be an arbitrary point. Define the n sequences $< X_{i_n} >$ of X_i, i = 1, 2..........n as follows:

$x_{1_n} = (\prod_1^n T_i)^n x_1$; $x_{2_n} = T_1\, x_{1_{n-1}}$; $x_{3_n} = T_2\, x_{2_n}$ .... $x_{n_n} = T_{n-1}\, x_{n-1_n}\ \forall\ n \in N$.

Assume that $x_{1_n} = x_{1_{n+1}}$, $x_{2_n} \neq x_{2_{n+1}}$ ....$x_{n_n} \neq x_{n_{n+1}}\ \forall\ n \in N$. Otherwise, if $x_{1_n} = x_{1_{n+1}}$ for some n, $x_{2_{n+1}} = x_{2_{n+2}}$, $x_{3_{n+1}} = x_{3_{n+2}}$ ....$x_{n_{n+1}} = x_{n_{n+2}}$ and we could put $x_{1_n} = \alpha_1$, $x_{2_{n+1}} = \alpha_2$, $x_{3_{n+1}} = \alpha_3$............$x_{n_{n+1}} = \alpha_n$. If $x_{2_n} = x_{2_{n+1}}$ then $x_{3_n} = x_{3_{n+1}}$, $x_{4_n} = x_{4_{n+1}}$.................$x_{n_n} = x_{n_{n+1}}$ and the later equalities imply that $x_{1_n} = x_{1_{n+1}}$. In a similar way if $x_{n-1_n} = x_{n-1_{n+1}}$ or $x_{n_n} = x_{n_{n+1}}$ then $x_{1_n} = x_{1_{n+1}}$.

Taking $x_3 = x_{3_{n-1}}$, $x_2 = x_{2_n}$ in (1.2) we obtain:

$$d_2(x_{2_n}, x_{2_{n+1}}) = d_2(\prod_3^1 T_i\ x_{3_{n-1}}, \prod_2^1 T_i\ x_{3_n}) \le c\ \frac{f_2(x_{2_n}, x_{3_{n-1}})}{g_2(x_{2_n}, x_{3_{n-1}})}$$

$$= \frac{c \max.\{d_2(x_{2_n}, x_{2_{n+1}})d_1(x_{1_{n-1}}, x_{1_n}); d_2(x_{2_n}, x_{2_{n+1}})d_n(x_{n_{n-1}}, x_{n_n}); d_2(x_{2_n}, x_{2_{n+1}})\, d_{n-1}(x_{n-1_{n-1}}, x_{n-1_n}) \ldots\ldots d_2(x_{2_n}, x_{2_{n+1}})d_3(x_{3_{n-1}}, x_{3_n}); d_2(x_{2_n}, x_{2_n})d_3(x_{3_{n-2}}, x_{3_n})\}}{\max.\{d_2(x_{2_n}, x_{2_n}); d_2(x_{2_n}, x_{2_{n+1}}); d_3(x_{3_n}, x_{3_n})\}}$$

$$= \frac{c \max.\{d_2(x_{2_n}, x_{2_{n+1}})[d_1(x_{1_{n-1}}, x_{1_n}); d_n(x_{n_{n-1}}, x_{n_n}); d_{n-1}(x_{n-1_{n-1}}, x_{n-1_n}) \ldots\ldots\ldots d_3(x_{3_{n-1}}, x_{3_n})]\}}{d_2(x_{2_n}, x_{2_{n+1}})}$$

$$= c \max.\{d_1(x_{1_{n-1}}, x_{1_n}); d_n(x_{n_{n-1}}, x_{n_n}); d_{n-1}(x_{n-1_{n-1}}, x_{n-1_n}) \ldots\ldots\ldots d_3(x_{3_{n-1}}, x_{3_n})\}$$

Thus $d_2(x_{2_n}, x_{2_{n+1}}) \le c \max.\{d_1(x_{1_{n-1}}, x_{1_n}); d_3(x_{3_{n-1}}, x_{3_n}); d_4(x_{4_{n-2}}, x_{4_n})$...................
$d_n(x_{n_{n-1}}, x_{n_n})\}$ (2.1)

Taking $x_4 = x_{4_{n-1}}$, $x_3 = x_{3_n}$ in (1.3) we obtain :

$$d_3(x_{3_n}, x_{3_{n+1}}) = d_3\left(\prod_4^2 T_i\, x_{4_{n-1}}, \prod_3^2 T_i\, x_{3_n}\right) \le c\,\frac{f_3(x_{3_n}, x_{4_{n-1}})}{g_3(x_{3_n}, x_{4_{n-1}})}$$

$$= c\,\frac{\max.\{d_3(x_{3_n}, x_{3_{n+1}})d_2(x_{2_n}, x_{2_{n+1}});\ d_3(x_{3_n}, x_{3_{n+1}})d_1(x_{l_{n-1}}, x_{l_n});\ d_3(x_{3_n}, x_{3_{n+1}})\, d_n(x_{n_{n-1}}, x_{n_n})\ldots\ldots d_3(x_{3_n}, x_{3_{n+1}})d_4(x_{4_{n-1}}, x_{4_n});\ d_3(x_{3_n}, x_{3_n})d_4(x_{4_{n-1}}, x_{4_n})\}}{\max.\{d_3(x_{3_n}, x_{3_n});\ d_3(x_{3_n}, x_{3_{n+1}});\ d_4(x_{4_n}, x_{4_n})\}}$$

$$= c\,\frac{\max.\{d_3(x_{3_n}, x_{3_{n+1}})[d_2(x_{2_n}, x_{2_{n+1}});\ d_1(x_{l_{n-1}}, x_{l_n});\ d_n(x_{n_{n-1}}, x_{n_n})\ldots\ldots d_4(x_{4_{n-1}}, x_{4_n})]\}}{d_3(x_{3_n}, x_{3_{n+1}})}$$

$d_3(x_{3_n}, x_{3_{n+1}}) \le c\max.\{d_2(x_{2_n}, x_{2_{n+1}});\ d_1(x_{1_{n-1}}, x_{1_n});\ d_4(x_{4_{n-1}}, x_{4_n});\ \ldots\ldots\ d_n(x_{n_{n-1}}, x_{n_n})\}$

Using (2.1) we get

$$d_3(x_{3_n}, x_{3_{n+1}}) \le c\max.\{d_1(x_{1_{n-1}}, x_{1_n});\ d_3(x_{3_n}, x_{3_{n+1}});\ d_4(x_{4_{n-1}}, x_{4_n})\ldots\ldots d_n(x_{n_{n-1}}, x_{n_n})\} \quad (2.2)$$

Continuing this process we obtain:

$$d_4(x_{4_n}, x_{4_{n+1}}) \le c\max.\{d_1(x_{1_{n-1}}, x_{1_n});\ d_3(x_{3_{n-1}}, x_{3_n});\ d_4(x_{4_{n-1}}, x_{4_n})\ldots\ldots d_n(x_{n_{n-1}}, x_{n_n})\} \quad (2.3)$$

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$$d_n(x_{n_n}, x_{n_{n+1}}) \le c\max.\{d_1(x_{1_{n-1}}, x_{1_n});\ d_3(x_{3_{n-1}}, x_{3_n});\ d_4(x_{4_{n-1}}, x_{4_n})\ldots\ldots d_n(x_{n_{n-1}}, x_{n_n})\} \quad (2.n\text{-}1)$$

Taking $x_1 = x_{2_n}$, $x_2 = x_{2_n}$ in (1.1) we obtain

$$d_1(x_{1_n}, x_{1_{n+1}}) = d_1\left(\prod_2^n T_i\, x_{2_n}, \prod_1^n T_i\, x_{1_n}\right) \le c\,\frac{f_1(x_{1_n}, x_{2_n})}{g_1(x_{1_n}, x_{2_n})}$$

$$= c\,\frac{\max.\{d_1(x_{1_n}, x_{1_{n+1}})d_n(x_{n_n}, x_{n_{n+1}});\ d_1(x_{1_n}, x_{1_{n+1}})d_{n-1}(x_{n-1_n}, x_{n-1_{n+1}});\ d_1(x_{1_n}, x_{1_{n+1}})\, d_{n-2}(x_{n-2_n}, x_{n-2_{n+1}})\ldots\ldots\ldots d_1(x_{1_n}, x_{1_{n+1}})d_2(x_{2_n}, x_{2_{n+1}});\ d_1(x_{1_n}, x_{1_n})d_2(x_{2_n}, x_{2_{n+1}})\}}{\max.\{d_1(x_{1_n}, x_{1_n});\ d_1(x_{1_n}, x_{1_{n+1}});\ d_2(x_{2_{n+1}}, x_{2_{n+1}})\}}$$

$$= c\,\frac{\max.\{d_1(x_{1_n}, x_{1_{n+1}})[d_n(x_{n_n}, x_{n_{n+1}});\ d_{n-1}(x_{n-1_n}, x_{n-1_{n+1}})\ \ldots\ldots\ldots\ldots d_2(x_{2_n}, x_{2_{n+1}})]\}}{d_1(x_{1_n}, x_{1_{n+1}})}$$

$d_1(x_{1_n}, x_{1_{n+1}}) = c\max.\{d_n(x_{n_n}, x_{n_{n+1}});d_{n-1}(x_{n-1_n}, x_{n-1_{n+1}})\ldots\ldots\ldots\ldots d_2(x_{2_n}, x_{2_{n+1}})\}$

Using (2.1), (2.2)...................... (2.n-1) we get

$$d_1(x_{1_n}, x_{1_{n+1}}) \le c\max.\{d_1(x_{1_{n-1}}, x_{1_n});d_3(x_{3_{n-1}}, x_{3_n})\ldots\ldots\ldots\ldots d_n(x_{n_{n-1}}, x_{n_n}) \quad (2.n)$$

Continuing this process by induction on inequalities (2.1), (2.2).......... (2.n) we get the following inequalities

$d_1(x_{1_n}, x_{1_{n+1}}) \le c^{n-1}\{d_1(x_{1_1}, x_{1_2}); d_3(x_{3_1}, x_{3_2}); d_4(x_{4_1}, x_{4_2}) \ldots\ldots d_n(x_{n_1}, x_{n_2})\}$.

$d_2(x_{2_n}, x_{2_{n+1}}) \le c^{n-1}\{d_1(x_{1_1}, x_{1_2}); d_3(x_{3_1}, x_{3_2}); d_4(x_{4_1}, x_{4_2}) \ldots\ldots d_n(x_{n_1}, x_{n_2})\}$.

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$d_n(x_{n_n}, x_{n_{n+1}}) \le c^{n-1}\{d_1(x_{1_1}, x_{1_2}); d_3(x_{3_1}, x_{3_2}); d_4(x_{4_1}, x_{4_2}) \ldots\ldots d_n(x_{n_1}, x_{n_2})\}$.

Since $0 \le c < 1$, the sequence $< x_{i_n} >$, i=1, 2....n are Cauchy sequences. Again since each of (X_i, d_i), i = 1, 2.........n is complete metric space so, $\lim_{n\to\infty} x_{i_n} = \alpha_i \;\; \forall$ i = 1, 2....n. Taking $x_1 = x_{1_n}$, $x_2 = \alpha_2$ in (1.1) we obtain[2,3] :

$$d_1\Big(\prod_2^n T_i\,\alpha_2 , x_{1_{n+1}}\Big) = d_1\Big(\prod_2^n T_i\,\alpha_2, \prod_1^n T_i\,x_{1_n}\Big) \le c\,\frac{f_1(x_{1_n},\alpha_2)}{g_1(x_{1_n},\alpha_2)}$$

$$= c\,\frac{\max.\Big\{d_1(x_{1_n}, x_{1_{n+1}})d_{2n}\Big(\prod_2^{n-1} T_i\,x_2, x_{n_{n+1}}\Big);\, d_1(x_{1_n}, x_{1_{n+1}})d_{n-1}\Big(\prod_2^{n-2} T_i\,x_2, x_{n-1_{n+1}}\Big);\ldots\ldots\ldots\; d_1(x_{1_n}, x_{1_{n+1}})d_2\Big(\alpha_2, \prod_2^1 T_i\,\alpha_2\Big);\, d_1\Big(x_{1_n}, \prod_2^n T_i\,\alpha_2\Big)d_2(\alpha_2, x_{2_{n+1}})\Big\}}{\max.\Big\{d_1\Big(x_{1_n}, \prod_2^n T_i\,\alpha_2\Big);\, d_1(x_{1_n}, x_{1_{n+1}});\, d_2\Big(x_{2_{n+1}}, \prod_2^1 T_i\,\alpha_2\Big)\Big\}}$$

Letting $n \to \infty$ we get $d_1\Big(\prod_2^n T_i\,\alpha_2 , \alpha_1\Big) \le 0$ from which it follows that $\prod_2^n T_i\,\alpha_2 = \alpha_1$. In a similar way, using the inequalities (1.2), (1.3)......... (1.n) it can be shown that

$$\prod_1^{n-1} T_i\,\alpha_1 = \alpha_n,\; \prod_n^{n-2} T_i\,\alpha_n = \alpha_{n-1} \ldots\ldots \prod_3^1 T_i\,\alpha_3 = \alpha_2.$$

Taking $x_3 = T_2\alpha_2$ and $x_2 = x_{2_n}$ in (1.2) we obtain:

$$d_2\Big(\prod_2^1 T_i\,\alpha_2 , x_{2_{n+1}}\Big) = d_2\Big(\prod_2^1 T_i\,\alpha_2, \prod_2^1 T_i\,x_{2_n}\Big) \le c\,\frac{f_2(x_{2_n},T_2\alpha_2)}{g_2(x_{2_n},T_2\alpha_2)}$$

$$= c\,\frac{\max.\Big\{d_2(x_{2_n}, x_{2_{n+1}})d_1\Big(\prod_2^n T_i\,x_2, x_{1_n}\Big);\, d_2(x_{2_n}, x_{2_{n+1}})d_n\Big(\prod_2^{n-1} T_i\,\alpha_2, x_{n_n}\Big)\ldots\ldots\ldots\; d_2(x_{2_n}, x_{2_{n+1}})d_3\Big(T_2\alpha_2, \prod_2^2 T_i\,\alpha_2\Big);\, d_2\Big(x_{2_n}, \prod_2^1 T_i\,\alpha_2\Big)d_3(T_2\alpha_2, x_{3_n})\Big\}}{\max.\Big\{d_2\Big(x_{2_n}, \prod_2^1 T_i\,\alpha_2\Big);\, d_2(x_{2_n}, x_{2_{n+1}});\, d_3\Big(x_{3_n}, \prod_2^2 T_i\,\alpha_2\Big)\Big\}}$$

Letting $n \to \propto$ and since $\prod_{2}^{n} T_i \ \alpha_2 = \alpha_1$ we get

$$d_2\Big(\prod_{2}^{1} T_i \ \alpha_2, \alpha_2\Big) \le c \ \frac{d_2\big(\alpha_2, \prod_{2}^{1} T_i \ \alpha_2\big) d_3(T_2\alpha_2, \ \alpha_3)}{\max.\{d_2\big(\alpha_2, \prod_{2}^{1} T_i\alpha_2\big); d_3(\alpha_3, \ T_2T_1\alpha_1)\}}$$

Here two cases arise:

Case (i) If $\max.\{d_2(\alpha_2, \prod_{2}^{1} T_i \ \alpha_2); \ d_3(\alpha_3, \ T_2T_1\alpha_1)\}$

$= d_2(\alpha_2, \prod_{2}^{1} T_i\alpha_2)$ we have

$$d_2\Big(\prod_{2}^{1} T_i\alpha_2, \ \alpha_2\Big) \le c \ \frac{d_2\big(\alpha_2, \prod_{2}^{1} T_i\alpha_2\big) d_3(T_2\alpha_2, \ \alpha_3)}{d_2\big(\alpha_2, \prod_{2}^{1} T_i\alpha_2\big)}$$

$$= c \ d_3(T_2\alpha_2, \ \alpha_3)$$

Case (ii) If $\max.\{d_2(\alpha_2, \prod_{2}^{1} T_i\alpha_2); \ d_3(\alpha_3, \ T_2T_1\alpha_1)\}$

$= d_3(\alpha_3, \ T_2T_1\alpha_1)$ and $d_2(\alpha_2, \prod_{2}^{1} T_i\alpha_2) \neq 0$

We have $d_2(\prod_{2}^{1} T_i \ \alpha_2, \ \alpha_2) = d_2(T_1\alpha_1, \ \alpha_2) \ \le$

$$\frac{c \ d_2\big(\alpha_2, \prod_{2}^{1} T_i\alpha_2\big) d_3(T_2\alpha_2 \ , \ \alpha_3)}{d_3(\alpha_3, \ T_2T_1\alpha_1)}$$

$$= c \ d_3(T_2\alpha_2, \ \alpha_3)$$

Thus in both cases we obtain:

$$d_2\Big(\prod_{2}^{1} T_i\alpha_2, \ \alpha_2\Big) = d_2(T_1\alpha_1, \ \alpha_2) \le c \ d_3(T_2\alpha_2, \ \alpha_3) \quad (3.1)$$

In the similar way using (1.3), (1.4)................ (1.n-1) and (1.1) it can be shown that

$$d_3\Big(\prod_{3}^{2} T_i \ \alpha_3, \ \alpha_3\Big) = d_3(T_2\alpha_2, \ \alpha_3) \le c \ d_4(T_3\alpha_3, \ \alpha_4) \quad (3.2)$$

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$$d_n\Big(\prod_{n}^{n-1} T_i \ \alpha_n, \ \alpha_n\Big) = d_n(T_{n-1}\alpha_{n-1}, \ \alpha_n) \le c \ d_{n+1}(T_n\alpha_n, \ \alpha_1) \quad (3.n\text{-}1)$$

$$d_1\Big(\prod_{1}^{n} T_i \ \alpha_1 \ , \ \alpha_1\Big) = d_2(T_n\alpha_n, \ \alpha_1) \le c \ d_2(T_1\alpha_1, \ \alpha_2) \quad (3.n)$$

Using (3.1), (3.2).................. (3.n) we obtain:

$$d_2\Big(\prod_{2}^{1} T_i\alpha_2, \ \alpha_2\Big) = d_2(T_1\alpha_1, \ \alpha_2) \le c \ d_3(T_2\alpha_2, \ \alpha_3)$$
$$\le \ c^2 \ d_4(T_3\alpha_3, \ \alpha_4) \le c^3 \ d_5(T_4\alpha_5, \ \alpha_5)..............$$
$$......\le c^n \ d_2(T_1\alpha_1, \ \alpha_2).$$

From which it follows $\prod_{2}^{1} T_i \ \alpha_2 = \alpha_2$, $T_2\alpha_2 = \alpha_3$; $T_3\alpha_3 = \alpha_4$..............$T_n\alpha_n = \alpha_1$, since $0 \le c < 1$.

Similarly we can show that $\prod_{3}^{2} T_i \ \ \alpha_3 = \alpha_3$, $\prod_{4}^{3} T_i \ \alpha_4 = \alpha_4$................ $\prod_{n}^{1} T_i \ \ \alpha_n = \alpha_n$ *i.e.* α_1, α_2.... ...... α_n are the fixed points of $\prod_{1}^{n} T_i$, $\prod_{2}^{1} T_i$ .... $\prod_{n}^{n-1} T_i$.

Now we show that these fixed points are unique. Let α_1' be the another fixed point of $\prod_{1}^{n} T_i$. Using (1.1) for $x_2 = T_1\alpha_1$ and $x_1 = \alpha_1'$ we get

$$d_1(\alpha_1, \alpha_1') = d_1\Big(\prod_{1}^{n} T_i \alpha_1, \prod_{1}^{n} T_i \alpha_1'\Big) \le c \ \frac{f_1(\alpha_1', T_1\alpha_1)}{g_1(\alpha_1', T_1\alpha_1)}$$

$$= c\ \frac{\max.\{d_1(\alpha_1',\alpha_1')d_n(\prod_1^{n-1}T_i\,\alpha_1, \prod_1^{n-1}T_i\,\alpha_1');\ d_1(\alpha_1,\alpha_1')d_{n-1}(\prod_1^{n-2}T_i\,\alpha_1, \prod_1^{n-2}T_i\,\alpha_1');\ldots\ldots\ldots\ d_1(\alpha_1',\alpha_1')d_2(T_1\alpha_1, \prod_1^{1}T_i\,\alpha_1);\ d_1(\alpha_1',\alpha_1)d_2(T_1\alpha_1, T_1\alpha_1')\}}{\max.\{d_1(\alpha_1', \alpha_1);\ d_1(\alpha_1', \alpha_1');\ d_2(T_1\alpha_1',T_1\alpha_1)\}}$$

$$= \frac{c\, d_1(\alpha_1',\alpha_1')\, d_2(T_1\alpha_1, T_1\alpha_1')}{\max.\{d_1(\alpha_1', \alpha_1);\ d_2(T_1\alpha_1',T_1\alpha_1)\}}$$

Here two cases arise;

Case (i) If max. $\{d_1(\alpha_1', \alpha_1);\ d_2(T_1\alpha_1',T_1\alpha_1)\} = d_2\,(T_1\alpha_1',T_1\alpha_1)$ then we get

$d_1(\alpha_1, \alpha_1') \le c\ d_1(\alpha_1', \alpha_1)$ which gives $\alpha_1' = \alpha_1$.

Case (ii) If max. $\{d_1(\alpha_1', \alpha_1);\ d_2(T_1\alpha_1',T_1\alpha_1)\} = d_1(\alpha_1', \alpha_1)$ then we get

$$d_1(\alpha_1 , \alpha_1') \le c\ d_2(T_1\alpha_1, T_1\alpha_1') \tag{4.1}$$

Now taking $x_3 = T_2T_1\alpha_1$ and $x_2 = T_1\alpha_1'$ in (1.2) we obtain:

$$d_2(T_1\alpha_1, T_1\alpha_1') = d_2(\prod_1^{1}T_i\,\alpha_1, \prod_1^{1}T_i\,\alpha_1') \le c\ \frac{f_2(T_1\alpha_1', T_2T_1\alpha_1)}{g_2(T_1\alpha_1', T_2T_1\alpha_1)}$$

$$= c\ \frac{\max.\{d_2(T_1\alpha_1',T_1\alpha_1')d_1(\alpha_1,\alpha_1');\ d_2(T_1\alpha_1',T_1\alpha_1')d_n(\prod_1^{n-1}T_i\,\alpha_1, \prod_1^{n-2}T_i\,\alpha_1');\ldots\ldots\ldots\ d_2(T_1\alpha_1',T_1\alpha_1')d_3(T_2T_1\alpha_1, \prod_1^{2}T_i\,\alpha_1);\ d_2(T_1\alpha_1',T_1\alpha_1)d_3(T_2T_1\alpha_1, T_2T_1\alpha_1')\}}{\max.\{d_2(T_1\alpha_1',T_1\alpha_1);\ d_2(T_1\alpha_1',T_1\alpha_1');\ d_3(T_2T_1\alpha_1',T_2T_1\alpha_1)\}}$$

$$= \frac{c\ d_2(T_1\alpha_1',T_1\alpha_1)d_3(T_2T_1\alpha_1, T_2T_1\alpha_1')}{\max.\ \{d_2(T_1\alpha_1',T_1\alpha_1);\ d_3(T_2T_1\alpha_1',T_2T_1\alpha_1)\}}$$

As discussed above we get

$$d_2(T_1\alpha_1, T_1\alpha_1') \le cd_3(T_2T_1\alpha_1, T_2T_1\alpha_1') \tag{4.2}$$

In a similar way using (1.3), (1.4).............. (1.n) it can be shown that

$$d_3(T_2T_1\alpha_1 , T_2T_1\alpha_1') \le cd_4(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1') \tag{4.3}$$

....

....

$$d_n(\prod_1^{n-1}T_i\,\alpha_1 , \prod_1^{n-1}T_i\,\alpha_1') \le cd_1(\prod_1^{n}T_i\,\alpha_1, \prod_1^{n}T_i\,\alpha_1') = c_1\, d_1(\alpha_1 , \alpha_1') \tag{4.n}$$

Using (4.1), (4.2)................ (4.n) we get

$d_1(\alpha_1, \alpha_1') \leq c\, d_2(T_1\alpha_1, T_1 1') \leq c^2\, d_3(T_2T_1\alpha_1, T_2T_1\alpha_1') \leq c^3\, d_4(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1') \leq c^n\, d_1(\alpha_1, \alpha_1')$

Thus $\alpha_1 = \alpha_1'$ *i. e.* α_1 is a unique fixed point of $\prod_1^n T_i$. In the same way we can show that α_2, α_3.............α_n be the unique fixed points of $\prod_2^1 T_i$, $\prod_3^2 T_i$ .... $\prod_n^1 T_i$ respectively.

This completes the proof of the theorem.

Remark:

i. If $X_4 = X_5 = X_6$...........$= X_n = X_1$, $d_4 = d_5 = d_6$.............$= d_n = d_1$ and the mappings $T_4 = T_5 = T_6$..........$= T_n$ as the identity mapping of X_1 then the theorem reduce to Jain[2].

ii. If $X_5 = X_6 = X_7$...........$= X_n = X_1$, $d_5 = d_4 = d_6 = d_7$.............$= d_n = d_1$ and the mappings $T_5 = T_6 = T_7$..........$= T_n$ as the identity mapping of X_1 then the theorem reduce to Luljeta and Kristaq[3].

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