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General Class of Inhomogeneous Perfect Magneto Fluid Solutions

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Abstract

We present a general class of inhomogeneous cosmological models filled with perfect magneto fluid by assuming that the background space-time admits two space-like commuting killing vectors and has separable metric coefficients. The singularity structure of these models depends on the choice of the parameters and the metric functions. A number of previously known perfect fluid models follow as particular cases of this general class. Physical and geometrical features of these models are studied.

1. Introduction

Senovilla[6] obtained a new class of exact solutions of Einstein's equations without big bang singularity representing a cylindrically symmetric, inhomogeneous universe filled with perfect fluid which is smooth and regular everywhere satisfying energy and causality conditions. Ruiz and Senovilla (1992) have separated out a large class of singularity free models through a comprehensive study of general cylindrically symmetric metric separable in r and t as metric coefficients[1-5].

In this paper we have presented an inhomogeneous metric admitting two space-like commuting killing vectors and separable metric coefficients and have obtained a general class of inhomogeneous solutions with perfect magnetofluid as the source term.

2. The Metric and Field Equations: We Start with the space-time admitting two commuting space-like killing vectors field which are hypersurface orthogonal. It is obvious that the metric of such space-time may be taken as the generalized Einstein-Rosen form. It is further assumed that the metric coefficients are separable in space-time. Ruiz and Senovilla (1992) have presented that the line-element for such space-time may be taken as[1-6]

$$ds^2 = T^{2m}F^2\left(dt^2 - H^2dx^2\right) - T^{n+1}GPdy^2 - T^{1-n}GP^{-1}dz^2\,, \tag{1}$$

where F,G,P,H are functions of only x, $T = T(t)$ and m and n are constants. The freedom of selecting coordinates is used in having the same time dependence in the metric coefficients

g_{11} and g_{00}. The co-ordinates are lebelled as

$$x^1 = x, x^2 = y, x^3 = z, x^0 = t. \tag{2}$$

Here the function $H=H(x)$ is introduced to facilitate the integration of Einstein field equations in terms of elementary functions.

Let us now introduce the orthonormal tetrad

$$\theta^1 = T^m FHdx, \qquad \theta^2 = T^{(1+n)/2}\sqrt{GP}dy$$
$$\theta^3 = T^{(1-n)/2}\sqrt{G/P}dz, \quad \theta^0 = T^m Fdt. \tag{3}$$

The non- vanishing components of Ricci tensor for the metric (1) in the above tetrad frame have the following expressions

$$R_{11}T^{2m}F^2 = \frac{1}{H^2}\left[\frac{G''}{G} - \frac{1}{2}\frac{G'^2}{G^2} + \frac{1}{2}\frac{P'^2}{P^2} - \frac{F''}{F} - \frac{F'G'}{FG} - \frac{H'G'}{HG} - \frac{F'^2}{F^2} - \frac{F'H'}{FH}\right] - m\frac{\ddot{T}}{T} \tag{4}$$

$$R_{22}T^{2m}F^2 = \frac{1}{H^2}\left[\frac{1}{2}\left(\frac{G''}{G} - \frac{P''}{P}\right) + \frac{1}{2}\frac{P'^2}{P^2} - \frac{1}{2}\frac{G'P'}{GP} - \frac{1}{2}\frac{H'G'}{HG} + \frac{1}{2}\frac{P'H'}{PH}\right] - \frac{1}{2}(1+n)\frac{\ddot{T}}{T} \tag{5}$$

$$R_{33}T^{2m}F^2 = \frac{1}{H^2}\left[\frac{1}{2}\left(\frac{G''}{G} + \frac{P''}{P}\right) - \frac{1}{2}\frac{P'^2}{P^2} + \frac{1}{2}\frac{G'P'}{GP} - \frac{1}{2}\frac{H'G'}{HG} - \frac{1}{2}\frac{P'H'}{PH}\right] - \frac{1}{2}(1-n)\frac{\ddot{T}}{T} \tag{6}$$

$$R_{00}T^{2m}F^2 = \frac{1}{H^2}\left[\frac{F''}{F} - \frac{F'^2}{F^2} - \frac{F'H'}{FH} + \frac{F'G'}{FG}\right] + (m+1)\frac{\ddot{T}}{T} + \frac{1}{2}\left(n^2 - 1 - 4m\right)\frac{\dot{T}^2}{T^2} \tag{7}$$

$$R_{10}T^{2m}F^2 = \frac{1}{2H}\frac{\dot{T}}{T}\left[(1-2m)\frac{G'}{G} + n\frac{P'}{P} - \frac{2F'}{F}\right] \tag{8}$$

Here a prime and a dot denote derivative with respect to x and t respectively.

The energy momentum tensor for perfect magnetofluid is taken as[7-10]

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta} + \tau_{\alpha\beta} \quad \text{where} \tag{9}$$

$$\tau_{\alpha\beta} = \mu\left[\left(u_\alpha u_\beta - \frac{1}{2}g_{\alpha\beta}\right)\left|\underset{\sim}{h}\right|^2 - h_\alpha h_\beta\right] \text{with} \tag{10}$$

$$u_\alpha h^\alpha = 0, \qquad \left|\underset{\sim}{h}\right|^2 = -h_\rho h^\rho \geq 0, \tag{11}$$

$$u_\alpha u^\alpha = 1. \tag{12}$$

According to equations (9) and (10), we get

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta} + \mu\left[\left(u_\alpha u_\beta - \frac{1}{2}g_{\alpha\beta}\right)\left|\underset{\sim}{h}\right|^2 - h_\alpha h_\beta\right]$$

The equation (13) taken the form

$$T_{\alpha\beta} = (\pi + w)u_\alpha u_\beta - \pi g_{\alpha\beta} - \mu h_\alpha h_\beta, \text{where} \quad (14)$$

$$\pi = p + \frac{1}{2}\mu \left|\underset{\sim}{h}\right|^2, \quad (15)$$

$$w = \rho + \frac{1}{2}\mu \left|\underset{\sim}{h}\right|^2. \quad (16)$$

Under magneto hydrodynamic approximation, the Maxwell equations are

$$\left(u^\alpha h^\beta - u^\beta h^\alpha\right)_{;\alpha} = 0, \quad (17)$$

In the co-moving coordinates, we get

$$u_\alpha = \sqrt{g_{00}}\delta_\alpha^0 = T^m F \delta_\alpha^0 \quad (18)$$

Here π and w denote the total pressure and the energy density of magnetofluid distribution. u_α denotes unit time like flow vector of fluid and h_α denotes the spacelike magnetic flow vector orthogonal to u_α. The components of these vectors in the tetrad frame are found to be[1-10]

$$u^\alpha = (0,0,0,1), \quad (19)$$

$$h_\alpha = (h,0,0,0), \quad (20)$$

Where h is a function of coordinates governed by the Einstein field equations

$$R_{\alpha\beta} = -x\left[(\pi + w)u_\alpha u_\beta - \frac{1}{2}g_{\alpha\beta}(w - \pi)\right] + x\mu h_\alpha h_\beta \quad (21)$$

Equations (14)-(20) imply the following relations,

$$R_{11} = \frac{1}{2}xg_{11}(w - \pi) + x\mu(h)^2, \quad (22)$$

$$R_{22} = \frac{1}{2}xg_{22}(w - \pi), \quad (23)$$

$$R_{33} = \frac{1}{2}xg_{33}(w - \pi), \quad (24)$$

$$R_{00} = -x\left[(w + \pi) - \frac{1}{2}g_{00}(w - \pi)\right] \quad (25)$$

$$R_{01} = 0 \quad (26)$$

In view of equation(8) we get, keeping $\dot{T} \neq 0$,

$$F^2 = G^{1-2m}P^n \quad (27)$$

Now from $R_{22} = R_{33,}$ one obtains

$$\frac{1}{H^2}\left(\frac{G'P'}{GP} + \frac{P''}{P} - \frac{P'^2}{P^2} - \frac{P'H'}{PH}\right) = -n\frac{\ddot{T}}{T}, \quad (28)$$

The right hand side of this equation is a function of t and left hand side is a function of x only. Hence, both of them must be equal to a separation constant. Therefore, we get

$$\frac{\ddot{T}}{T} = \varepsilon a^2, \quad \varepsilon = 0, \pm 1, \text{ a=constant, or equivalently} \quad (29)$$

$$T(t) = At + B \quad \text{for} \quad \varepsilon = 0, \quad (30)$$

$$T(t) = A\cosh(at) + B\sinh(at) \quad \text{for } \varepsilon = 1, \quad (31)$$

$$T(t) = A\cos(at) + B\sin(at) \quad \text{for } \varepsilon = -1, \quad (32)$$

Where A and B are arbitrary constants

of integration. Apart from this, from equation (28) one obtains

$$\frac{G'P'}{GP}+\frac{P''}{P}-\frac{P'^2}{P^2}-\frac{P'H'}{PH}=\varepsilon na^2H^2 \quad (33)$$

If R_{11} has to be equal R_{33}, then magnetic field must vanishes, and we get

$$(1-m)\left[\frac{G''}{G}-\frac{G'H'}{GH}\right]+\frac{4m-3}{2}\frac{G'^2}{G^2}-n\frac{G'P'}{GP}$$

$$+\frac{1}{2}\frac{P'^2}{P^2}=\left(\frac{2m-1-n^2}{2}\right)\varepsilon a^2H^2-\mu(h)^2. \quad (34)$$

In view of equations (18),(13),(4)and (7), one obtains

$$\chi W=T^{-2m}F^{-2}\left[\left(\frac{G'H'}{GH}-\frac{1}{4}\frac{P'^2}{P^2}+\frac{F'G'}{FG}\right.\right.$$

$$\left.\left.-\frac{G''}{G}+\frac{1}{4}\frac{G'^2}{G^2}\right)\frac{1}{H^2}+\frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right], \quad (35)$$

$$\chi\pi=T^{-2m}F^{-2}\left[\frac{1}{H^2}\left(\frac{F'G'}{FG}+\frac{1}{4}\frac{G'^2}{G^2}-\frac{1}{4}\frac{P'^2}{P^2}\right)\right.$$

$$\left.-\varepsilon a^2+\frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right], \quad (36)$$

Where χ is the gravitational constant.

In the limit of vanishing magnetic field, we recover the solutions of Ruiz and Senovilla (1992), For m=1,

$$F^2=\frac{P^n}{G} \quad (37)$$

Hence, we get the equation of state π=W, *i.e.* stiffmagneto fluid. Now we assume m$\neq$1. Let us define

$$\alpha=\frac{P'}{P}, \quad \mu=\frac{G'}{G} \quad (38)$$

In view of the equation(38), we can rewrite the equations (33) and (34) in following form

$$\alpha'+\alpha\beta-\alpha\frac{H'}{H}=\varepsilon na^2H^2 \quad (39)$$

$$\beta'+\frac{2m-1}{2(1-m)}\beta^2-\frac{n}{1-m}\beta\alpha+\frac{1}{2(1-m)}\alpha^2$$

$$-\beta\frac{H'}{H}=\frac{2m-1-n^2}{2(1-m)}\varepsilon a^2H^2-\mu(h)^2. \quad (40)$$

Where H may be taken as disposable function. Hence, the density and pressure take the form

$$\chi W=T^{-2m}F^{-2}\left[\frac{m+1}{m-1}\mu+\frac{4m+1-n^2}{4}\left(\frac{\dot{T}^2}{T^2}-\varepsilon a^2\right)\right], \quad (41)$$

$$\chi\pi=T^{-2m}F^{-2}\left[\mu+\frac{4m+1-n^2}{4}\left(\frac{\dot{T}^2}{T^2}-\varepsilon a^2\right)\right] \quad (42)$$

Where

$$\mu = \frac{1}{H^2}\left[\frac{n}{2}\alpha\beta - \tfrac{1}{4}\alpha^2 + \frac{3-4m}{4}\beta^2 + \frac{4m-3-n^2}{4}\varepsilon a^2 H^2\right]. \tag{43}$$

It follows from these expressions that the equation of state must be of the form $\pi=\pi(W)$ if

$$4m+1 = n^2, m \geq -4 \tag{44}$$

Or

$$T(t) = e^{at}, \varepsilon = 1. \tag{45}$$

Hence, we get

$$\pi = \gamma W, \tag{46}$$

where

$$\gamma = \frac{m-1}{m+1} \tag{47}$$

In a straightforward manner, one may evaluate the kinematical parameters *i.e.* expression for the expansion acceleration and shear of the fluid as[1-5]

$$\theta = \frac{1}{T^m F}\frac{\dot{T}}{T}(m+1), \tag{48}$$

$$a_1 = \frac{1}{T^m FH}\frac{F'}{F}, \tag{49}$$

$$\sigma_{11} = \frac{2m-1}{3}\frac{1}{T^m F}\frac{\dot{T}}{T}, \tag{50}$$

$$\sigma_{22} = \frac{1+3n-2m}{6}\frac{1}{T^m F}\frac{\dot{T}}{T}, \tag{51}$$

$$\sigma_{33} = \frac{1-3n-2m}{6}\frac{1}{T^m F}\frac{\dot{T}}{T}. \tag{52}$$

Where all other components as well as rotation tensor vanish.

3. *Solution of Maxwell Equations:*

From equation (17), we get in tetrad frame

$$\left(u^0 h^\beta\right)_{;0} + \left(-u^\beta h^1\right)_{;1} = 0 \tag{53}$$

Now for $\beta=0$

$$\left(u^0 h^1\right)_{,1} = 0 \tag{54}$$

for $\beta=1$

$$\left(u^0 h^1\right)_{,0} = 0 \tag{55}$$

For $\beta=2$ and $\beta=3$, the equation (53) is automatically satisfied. In view of equation (54) it is obvious that magnetic fluid h is not function of x and t.

4. **Concluding Remarks**

We have presented the generalization of the solutions of Ruiz and Senovilla (1992). In the limit of vanishing magnetic field we recover the above solutions. Some explicit solutions and a family of singularity free solutions are the next task and shall be presented in the next paper.

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