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Methods of Multi-Term Fractional Differential Equation

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Abstract

In this paper, we introduce a new algorithm, which produce a system of equations of low dimension, suitable for linear and non- linear Fractional Differential Equations (FDE).

1. Introduction

Multi-term equations have been introduced into the academic and scientific community with the aim of modeling a variety of physical behaviors. The first example put forward was by Bagley-Torvik[1] in modeling the motion of a rigid plate immersed in a Newtoian fluid. An algorithm tackling the numerical solution of this equation was introduced by Diethelm and Ford[2]. They use the least common multiple of the fractional and whole powers, in a similar manner to used to solve ordinary differential equations (ODEs) of order >1, to determine the size of the required system. The underlying principle from this algorithm was generalized in a subsequent paper[3], to enable approximate solutions of all equations of the form (2.1). But nothing seems to have been done to introduce an algorithm which is suitable for both linear and non-linear Fractional Differential Equations. Therefore, in this paper, for the first time we introduce an algorithm, which is suitable for both linear and non-linear Fractional Differential Equations.

2. *Definitions*

Multi-Term FDE

The non-linear multi-term FDE is defined as

$$D^{\alpha N} y(t) = f\left(t,\, y(t),\, D^{\alpha 1} y(t), \ldots\ldots, D^{\alpha\, N-1} y(t)\right) \tag{2.1}$$

equipped with initial conditions,

$$y^{(k)}(0) = y_0^{(k)},\ k = 0, 1, \ldots\ldots, \lceil \alpha_N \rceil - 1. \tag{2.2}$$

The linear multi-term FDE is defined as

$$\left[D^{\alpha N} + b_{N-1} D^{\alpha\, N-1} + \ldots\ldots + b_1 D^{\alpha 1} + b_0 D^0\right] y(t) = g(t), \tag{2.3}$$

with initial conditions (1.2).

General Multi-term Equations :

More generally, for the equation,

$$D^2 y = f\left(t, y(t), D^{\alpha_1} y(t), ...D^{\alpha\, m} y(t),\right.$$
$$\left. Dy(t), D^{\beta_1+1} y(t), ..., D^{1+\beta_{N-m}} y(t)\right), \quad (2.4)$$
$$where\ \alpha,\ \beta\ \in (0,1),$$

subject to initial conditions,

$$y^{(k)}(0) = y_0^{(k)},\ k = 0, 1$$

3. *Main Theorems :*

Now we give the following new theorems:

Theorem 3.1. The equation (2.4), with the relevant initial conditions, is equivalent to the system of equations

$$\begin{aligned}
{}^1Y(t) &= y(t) & & \\
{}^2Y(t) &= D^{\alpha_1} y(t) & &= D^{\alpha_1}\, {}^1Y(t) \\
{}^3Y(t) &= Dy(t) & &= D^{1-\alpha_1}\, {}^2Y(t) \\
&\vdots & & \\
{}^{m+1}Y(t) &= D^{\alpha\, m} y(t) & &= D^{\alpha_m - \alpha_{m-1}}\, {}^mY(t) \\
{}^{m+2}Y(t) &= Dy(t) & &= D^{1-\alpha_m}\, {}^{m+1}Y(t) \qquad (3.1) \\
{}^{m+3}Y(t) &= D^{\beta_1+1} y(t) & &= D^{\beta}\, {}^{m+2}Y(t) \\
&\vdots & & \\
{}^{N+1}Y(t) &= D^{\beta_{N-M}} y(t) & &= D^{\beta_{N-m} - \beta_{N-m-1}}\, {}^NY(t) \\
{}^{N+2}Y(t) &= D^2 y(t) & &= D^{1-\beta_{N-m}}\, {}^{N+1}Y(t)
\end{aligned}$$

together with initial conditions

$${}^kY(0) = \begin{cases} y_0^{(k)} & for\ k = 0\ and\ k = m+1, \\ 0 & else. \end{cases} \qquad (3.2)$$

Expressed in matrix form gives

$$\begin{pmatrix} D^{\alpha_1} & & 0 & \\ & D^{\alpha_2-\alpha_1} \ldots D^{1-\alpha_m} & & \\ & & D^{\beta_1} & \\ 0 & & & D^{\beta_2-\beta_1} .....D^{2-\beta_{N-m}} \end{pmatrix} \begin{pmatrix} {}^1Y(t) \\ {}^2Y(t) \\ \vdots \\ {}^{N+2}Y(t) \end{pmatrix} = \begin{pmatrix} {}^1S \\ {}^2S \\ \vdots \\ {}^{N+2}S \end{pmatrix} \qquad (3.3)$$

where

$$\begin{pmatrix} {}^1S \\ {}^2S \\ \vdots \\ {}^{N+2}S \end{pmatrix} = \begin{pmatrix} {}^2Y(t) \\ {}^3Y(t) \\ \vdots \\ f\left(t,\ {}^0Y(t),\ {}^1Y(t),\ {}^2Y(t), ...,\ {}^{N+1}Y(t)\right) \end{pmatrix}$$

Theorem 3.2. Assuming that the solution y of the initial value problem (2.4) is such that:
PREDICTOR

$$\left| \int_0^{t_{k+1}} (t_{k+1} - x)^{\alpha-1} D^{\alpha} y(x)\, dx - \sum_{j=0}^{k} b_{j,k+1} D^{\alpha} y(t_j) \right| \le C_1 t_{k+1}^{\gamma_1} h^{\delta_1} \qquad (3.2.1)$$

CORRECTOR

$$\left|\int_0^{t_{k+1}} \left(t_{k+1}-x\right)^{\alpha-1} D^{\alpha} y(x)\, dx - \sum_{j=0}^{k} b_{j,k+1} D^{\alpha} y\left(t_j\right)\right| \le S_1 t_{k+1}^{\gamma_2} h^{\delta_2} \tag{3.2.2}$$

with some $\gamma_1, \gamma_2 \ge 0$ and $\delta_1, \delta_2 \ge 0$. Then we choose $T > 0$, we have

$$\max_{0 \le j \le N} \left|y\left(t_j\right) - y_j\right| = O\left(h^q\right) \tag{3.2.3}$$

where $q = \min\{\delta_1 + \alpha, \delta_2\}$ and $N = \lfloor T/h \rfloor$.

4. *Proof of TheTheorems*

Proof of Theorem 3.1 :

The proof follows as a direct result of the commutable property of the Caputo derivative[4]. *i.e.*

$$D^{\alpha} D^{\beta} y(t) = D^{\beta} D^{\alpha} y(t) = D^{\alpha+\beta} y(t), \quad \forall\ \alpha, \beta \in N_{+}.$$

Example:

Our first example considers a non-linear equation with multiple β powers, exact solution of $y = t^3$.
The equation

$$D^2 y + t^2 D^{1.8} y + 4t D^{1.5} y + 5y^2 = 6t + \frac{25t^{2.5}}{\Gamma(.2)} + \frac{32t^{2.5}}{\Gamma(.5)} + 5t^6,$$

subject to the initial conditions,

$$y(0) = y'(0) = 0,$$

can be written as

$$\begin{pmatrix} D^{1/2} & 0 & 0 & 0 & 0 \\ 0 & D^{1/2} & 0 & 0 & 0 \\ 0 & 0 & D^{1/2} & 0 & 0 \\ 0 & 0 & 0 & D^{0.3} & 0 \\ 0 & 0 & 0 & 0 & D^{0.2} \end{pmatrix} \begin{pmatrix} {}^0Y \\ {}^1Y \\ {}^2Y \\ {}^3Y \\ {}^4Y \end{pmatrix} = \begin{pmatrix} {}^1S \\ {}^2S \\ {}^3S \\ {}^4S \\ {}^5S \end{pmatrix},$$

where

$$\begin{pmatrix} {}^1S \\ {}^2S \\ {}^3S \\ {}^4S \\ {}^5S \end{pmatrix} = \begin{pmatrix} {}^1Y(t) \\ {}^2Y(t) \\ {}^3Y(t) \\ {}^4Y(t) \\ 6t + \frac{25t^{3.2}}{\Gamma(.2)} + \frac{32t^{2.5}}{\Gamma(.5)} + 5t^6 - 5\left({}^0Y\right)^2 - 4t\left({}^3Y\right) - t^2\left({}^4Y\right) \end{pmatrix}$$

Note that even though there is no derivative between 0 and 1, we include a $D^{1/2}$ derivative, but with zero coefficients.

Proof of Theorem 3.2 :

Our proof is based upon the proof on lemma 3.1 of [5]. For h sufficiently small, we show that

$$\left|y\left(t_j\right) - y_j\right| \le Ch^q \tag{4.1}$$

$\forall\ j \in \{0, 1, 2, \ldots\ldots, n\}$, where C is a constant of suitable size.Mathematical induction will be the basis of the proof. As the initial conditions are given, the induction basis $(j = 0)$ is accepted. As in[5], (4.1) is now assumed true for $j = 0, 1, \ldots., k$ and some $k \le n-1$. We now prove that the inequality holds for $j = k+1$. We first concentrate on the behavior of the predictor.

Recall that, after the application of the first sorder Ψ_1, we have

$$\left| y(t_{k+1}) - y_{k+1}^{P} \right| = \frac{1}{\Gamma(\Psi_1)} \left| \int_0^{t_{k+1}} (t_{k+1} - x)^{\Psi_1 - 1} f(t, y(t))\, dt - \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j) \right|$$

$$\leq \frac{1}{\Gamma(\Psi_1)} \left| \int_0^{t_{k+1}} (t_{k+1} - x)^{\Psi_1 - 1} D^{\Psi_1} y(x)\, dx - \sum_{j=0}^{k} b_{j,k+1} D^{\Psi_1} y(t_j) \right|$$

$$+ \frac{1}{\Gamma(\Psi_1)} \sum_{j=0}^{k} b_{j,k+1} \left| f(t_j, y(t_j)) - f(t_j, y_j) \right|$$

$$\leq \frac{C_1 t_{k+1}^{\gamma_1}}{\Gamma(\Psi_1)} h^{\delta_1} + \frac{1}{\Gamma(\Psi_1)} \sum_{j=0}^{k} b_{j,k+1} L C h^{q}$$

$$\leq \frac{C_1 T^{\gamma_1}}{\Gamma(\Psi_1)} h^{\delta_1} + \frac{C L T^{\Psi_1}}{\Gamma(\Psi_1 + 1)} h^{q} \tag{4.2}$$

We then repeat for Ψ_2, thus

$$\left| y(t_{k+1}) - y_{k+1}^{P} \right| = \frac{1}{\Gamma(\Psi_2)} \left| \int_0^{t_{k+1}} (t_{k+1} - x)^{\Psi_2 - 1} f(t, y(t))\, dt - \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j) \right|$$

$$\leq \frac{1}{\Gamma(\Psi_2)} \left| \int_0^{t_{k+1}} (t_{k+1} - x)^{\Psi_2 - 1} D^{\Psi_2} y(x)\, dx - \sum_{j=0}^{k} b_{j,k+1} D^{\Psi_2} y(t_j) \right|$$

$$+ \frac{1}{\Gamma(\Psi_2)} \sum_{j=0}^{k} b_{j,k+1} \left| f(t_j, y(t_j)) - f(t_j, y_j) \right|$$

Now substituting as before, but now including the first predictor error bound (4.2), we obtain the new error bound,

$$\left| y(t_{k+1}) - y_{k+1}^{P} \right| \leq \frac{C_2 t_{k+1}^{\gamma_2}}{\Gamma(\Psi_2)} h^{\delta_1} + \frac{1}{\Gamma(\Psi_2)} \sum_{j=0}^{k} b_{j,k+1} L \left(\frac{C_1 T^{\gamma_1}}{\Gamma(\Psi_1)} h^{\delta_1} + \frac{C L T^{\Psi_1}}{\Gamma(\Psi_1 + 1)} h^{q} \right)$$

$$\leq \frac{C_2 T^{\gamma_2}}{\Gamma(\Psi_2)} h^{\delta_1} + \frac{L T^{\Psi_2}}{\Gamma(\Psi_2 + 1)} \left(\frac{C_1 T^{\gamma_1}}{\Gamma(\Psi_1)} h^{\delta_1} + \frac{C L T^{\Psi_1}}{\Gamma(\Psi_1 + 1)} h^{q} \right)$$

$$\leq \frac{C_2 T^{\gamma_2}}{\Gamma(\Psi_2)} h^{\delta_1} + \frac{C_1 L T^{\gamma_1} T^{\Psi_2}}{\Gamma(\Psi_2+1)\Gamma(\Psi_1)} h^{\delta_1} + \frac{C L^2 T^{\Psi_1} T^{\Psi_2}}{\Gamma(\Psi_2+1)\Gamma(\Psi_1+1)} h^q \quad .$$

This process is continued sequentially for all Ψ s in the system, thus obtaining an error bound for the predictorof,

$$\left|y(t_{k+1}) - y_{k+1}^P\right| = \left(\frac{C_n T^{\gamma_n}}{\Gamma(\Psi_n)} + \frac{C_{n-1} L T^{\gamma_{n-1}} T^{\Psi_n}}{\Gamma(\Psi_n+1)\Gamma(\Psi_{n-1})} + \ldots\ldots + \frac{C_1 L^{n-1} T^{\gamma_1} T^{\Psi_n} \ldots . T^{\Psi_2}}{\Gamma(\Psi_n+1) \ldots\ldots\ldots \Gamma(\Psi_2+1)\Gamma(\Psi_1)} \right) h^{\delta_1}$$

$$+ \left(\frac{C L^n T^{\gamma_1} T^{\Psi_n} \ldots\ldots T^{\Psi_1}}{\Gamma(\Psi_n+1) \ldots\ldots\ldots \Gamma(\Psi_1+1)} \right) h^q \tag{4.3}$$

By choosing T sufficiently small, we can now bound the terms of the first parentheses by C_{p_1} and the terms of the second parentheses by $C_{p_2} C$, thus for the predictor

$$\left|y(t_{k+1}) - y_{k+1}^P\right| \leq C_{p_1} h^{\delta_1} + C_{p_2} C h^q \quad . \tag{4.4}$$

We now use the error bound (4.3) in the analysis of the corrector.
Similar to above we have,

$$\left|y(t_{k+1}) - y_{k+1}\right| = \frac{1}{\Gamma(\Psi_1)} \left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\Psi_1 - 1} f(t, y(t))\, dt - \sum_{j=0}^{k+1} a_{j,k+1} f(t_j, y_j) \right|$$

$$\leq \frac{1}{\Gamma(\Psi_1)} \left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\Psi_1 - 1} D^{\Psi_1} y(t)\, dt - \sum_{j=0}^{k+1} a_{j,k+1} D^{\Psi_1} y(t_j) \right|$$

$$+ \frac{1}{\Gamma(\Psi_1)} \sum_{j=0}^{k} a_{j,k+1} \left| f(t_j, y(t_j)) - f(t_j, y_j) \right|$$

$$+ \frac{1}{\Gamma(\Psi_1)} a_{k+1,k+1} \left| f(t_{k+1}, y(t_{k+1})) - f(t_{k+1}, y_{k+1}^P) \right| \tag{4.5}$$

$$\leq \frac{S_1 t_{k+1}^{\varsigma_1}}{\Gamma(\Psi_1)} h^{\delta_2} + \frac{S L h^q}{\Gamma(\Psi_1)} \sum_{j=0}^{k} a_{j,k+1} + a_{k+1,k+1} \frac{L}{\Gamma(\Psi_1)} \left(C_{p_1} h^{\delta_1} + C_{p_2} h^q \right)$$

$$\leq \left(\frac{S_1 T^{\varsigma_1}}{\Gamma(\Psi_1)} + \frac{S L T^{\Psi_1}}{\Gamma(\Psi_1+1)} + \frac{C_{p_1} L}{\Gamma(\Psi_1+2)} + \frac{C_{p_2} C L}{\Gamma(\Psi_1+2)} h^{\Psi_1} \right) h^q \quad .$$

We repeat for Ψ_2 and substitute (4.5), we have

$$\left|y(t_{k+1}) - y_{k+1}\right| \le \frac{S_2 t_{k+1}^{\varsigma_2}}{\Gamma(\Psi_2)} h^{\delta_2} + \frac{SLh^q}{\Gamma(\Psi_2)} \sum_{j=0}^{k} a_{j,\,k+1}$$

$$+ a_{k+1,\,k+1} \frac{L}{\Gamma(\Psi_2)} \left(\frac{S_1 T^{\varsigma_1}}{\Gamma(\Psi_1)} + \frac{SLT^{\Psi_1}}{\Gamma(\Psi_1+1)} + \frac{C_{p_1} L}{\Gamma(\Psi_1+2)} + \frac{C_{p_2} CL}{\Gamma(\Psi_1+2)} h^{\Psi_1} \right) h^q$$

This process is continued sequentially for all Ψ s in the system, and then by choosing a sufficiently small T and a sufficiently large C, we can bound each summed in the parentheses and thus obtain an entire upper bound of Ch^q.

5. Conclusion

In this paper we have introduced the reader to the concept of multi-term FDEs. We then introduced a new method, specifically designed to optimize numerical efficiency for non-linear equations but which also very efficient for most linear equations. In the next chapter we now explore and compare the numerical efficiency of these algorithms and give recommendations of the most suitable algorithm for any particular problem.

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