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On Simultaneous Approximation for Baskakov-Durrmeyer-Stancu type operators

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Abstract

The aim of the present paper is to introduce and study the mixed summation-integral type operators having Baskakov and Beta basis functions in summation and integration, respectively. First, we estimate moments of these operators using hypergeometric series. Next, we obtain the basic point-wise convergence, a Voronovskaja type asymptotic formula.

Key words: Hypergeometric series, Baskakov-Durrmeyer-Stancu operators, Voronvoskaja type asymptotic formula.

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1. Introduction

In the year 1998, Agrawal and Thamer[1] introduced a new sequence of linear positive operators to study the simultaneous approximation of unbounded functions. For $f \in [0,\infty)$, a new type of Baskakov-Durrmeyer operator studied by Finta[2] is defined as

$$D_n(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)\, f(t)\,dt + p_{n,0}(x) f(0), \tag{1.1}$$

where $b_{n,k}(x) = \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}}$ and

$$b_{n,k}(t) = \frac{(n+1)_k (n+2)_k}{k!} \frac{t^k}{(1+t)^{n+k+2}}.$$

The Pochhammer symbol $(n)_k$ is defined as $(n)_k = n(n+1)(n+2)(n+3)....(n+k-1)$.

Recently, Gupta *et al.*[4] introduced Baskakov-Durrmeyer operators and investigated properties like point-wise convergence, asymptotic formula and inverse result in simultaneous approximation. Also, Govil and Gupta[3] studied some approximation properties for the operators defined in (1.1) and estimated local results in terms of modulus of continuity. After that, Ismail and Simeonov[8] introduced Positive linear integral operators in hypergeometric form, we write the operators (1.1) as

$$D_n(f,x)=(n+1)\sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\int_0^{\infty}\frac{(n+2)_k}{k!}\frac{t^k}{(1+t)^{n+k+2}}f(t)dt+\frac{f(0)}{(1+x)^n}$$

$$=(n+1)\int_0^{\infty}\frac{f(t)(1+x)^2}{[(1+x)(1+t)]^{n+2}}\sum_{k=1}^{\infty}\frac{(n)_k(n+2)_k}{(k!)^2}\frac{(xt)^k}{[(1+x)(1+t)]^k}dt+\frac{f(0)}{(1+x)^n}.$$

By hypergeometric series ${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}(x)^k$ and using the equality $(1)_k=k!$, we can write

$$D_n(f,x)=(n+1)\int_0^{\infty}\frac{f(t)(1+x)^2}{[(1+x)(1+t)]^{n+2}}\left({}_2F_1\left(n,n+2;1;\frac{xt}{(1+x)(1+t)}\right)-1\right)dt+\frac{f(0)}{(1+x)^n}.$$

Now using ${}_2F_1(a,b;c;x)={}_2F_1(b,a;c;x)$ and applying Pfaff- Kummer transformation

$${}_2F_1(a,b;c;x)=(1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$$

we have

$$D_n(f,x)=(n+1)\int_0^{\infty}f(t)(1+x)^2\left(\frac{{}_2F_1\left(n+2,1-n;1;\frac{-xt}{1+x+t}\right)}{(1+x+t)^{n+2}}-\frac{1}{[(1+x)(1+t)]^{n+2}}\right)dt+\frac{f(0)}{(1+x)^n}. \tag{1.2}$$

This is the form of the operators (1.1) in terms of hypergeometric functions.

Verma *et al.*[9] considered Baskakov -Durrmeyer- Stancu operators and studied some approximation properties of these operators. Very recently, Gupta and Yadav[5] introduced the Baskakov -Beta- Stancu operator and investigated like asymptotic formula, moments of these operators using hypergeometric series and errors estimation in simultaneous approximation. Hence, we introduce Baskakov -Durrmeyer- Stancu operators, for $0 \le \alpha \le \beta$ as

$$D_{n,\alpha,\beta}(f,x) = (n+1)\int_0^\infty f\left(\frac{nt+\alpha}{n+\beta}\right)(1+x)^2\left[\frac{{}_2F_1\left(n+2,1-n;1;\frac{-xt}{1+x+t}\right)}{(1+x+t)^{n+2}} - \frac{1}{[(1+x)(1+t)]^{n+2}}\right]dt$$

$$+\frac{f(0)}{(1+x)^n}. \qquad (1.3)$$

For $\alpha = \beta = 0$ the operators (1.3) reduces tothe operators (1.1).
We know that

$$\sum_{k=0}^{\infty} p_{n,k}(x) = 1,\ \int_0^\infty p_{n,k}(x)dx = \frac{1}{n-1},\ \sum_{k=1}^{\infty} b_{n,k}(t) = n+1,\ \int_0^\infty b_{n,k}(t)dt = 1.$$

We take

$$C_\nu[0,\infty)=\{f \in C[0,\infty) : f(t)=O(t)^\nu, \nu > 0\}.$$

The operators $D_{n,\alpha,\beta}(f, x)$ are well defined for $f \in C[0,\infty)$. In the present note, first, we establish the basic pointwise convergence theorem. Next, we also study asymptotic formula for these operators and estimate moments of Baskakov -Durrmeyer- Stancu operators using the techniques of hypergeometric series.

2. *Auxiliary results :*

In the sequel we shall need several lemmas.

Lemma 1. For $n > 0$ and $s > -1$, *we have*

$$D_n(t^s,x) = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}\left[(1+x)^s {}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right) - (1+x)^{-n}\right]. \quad (2.1)$$

Moreover,

$$D_n(t^s,x) = \frac{(n+s-1)!(n-s)!}{n!(n-1)!}x^s + \frac{s(s-1)(n+s-2)!(n-s)!}{n!(n-1)!}x^{s-1} + O(n^{-2}). \quad (2.2)$$

Proof. Taking $f(t) = t^s$, $t = (1+x)u$ and using Pfaff-Kummer transformation the right-hand side of (1.2), we get

$$D_n(t^s,x) = (n+1)\int_0^\infty \frac{(1+x)^{s+3}u^s}{[(1+x)(1+u)]^{n+2}}\sum_{k=0}^{\infty}\frac{(1-n)_k(n+2)_k}{(k!)^2}\frac{(-x(1+x)u^k}{[(1+x)(1+u)]^k}du$$

$$+\frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}(1+x)^{-n} = Q_1 + Q_2 \text{ (say)}.$$

$$Q_1 = (n+1)\sum_{k=0}^{\infty}\frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}\int_0^{\infty}\frac{u^{s+k}}{(1+u)^{n+k+2}}du$$

$$= (n+1)\sum_{k=0}^{\infty}\frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}B(s+k+1,n-s+1)$$

$$= (n+1)\sum_{k=0}^{\infty}\frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}\frac{\Gamma(s+k+1)\Gamma(n-s+1)}{\Gamma(n+K+2)}.$$

Using $\Gamma(n+k+2)=\Gamma(n+2)(n+2)_k$, we have

$$Q_1 = (n+1)\sum_{k=0}^{\infty}\frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}\frac{\Gamma(s+1)(s+1)_k\Gamma(n-s+1)}{\Gamma(n+2)(n+2)_k}$$

$$= (1+x)^{s-n+1}\frac{\Gamma(s+1)\Gamma(n-s+1)}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{(s+1)_k(1-n)_k}{(k!)^2}(-x)^k$$

$$= (1+x)^{s-n+1}\frac{\Gamma(s+1)\Gamma(n-s+1)}{\Gamma(n+1)}\,{}_2F_1(1-n,1+s;1;-x).$$

Using ${}_2F_1(a,b;c;x) = (1-x)^{-a}\,{}_2F_1\left(a, c-b;c;\frac{x}{x-1}\right)$, we have

$$Q_1 = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}(1+x)^s\,{}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right).$$

Combining Q_1 and Q_2 we get

$$D_n(t^s,x) = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}\left[(1+x)^s{}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right)-(1+x)^{-n}\right].$$

The other consequence (2.2) follows from the above equation by writing the expansion of hypergeometric series.

Lemma 2. For $0 \le \alpha \le \beta$ and m > 0 we have

$$D_{n,\alpha,\beta}(t^s,x)=x^s\frac{n^s}{(n+\beta)^s}\frac{(n+s-1)!(n-s)!}{n!(n-1)!}$$
$$+x^{s-1}\left\{s(s-1)\frac{n^s}{(n+\beta)^s}\frac{(n+s-2)!(n-s)!}{n!(n-1)!}+s\alpha\frac{n^{s-1}}{(n+\beta)^s}\frac{(n+s-2)!(n-s+1)!}{n!(n-1)!}\right\}$$
$$+x^{s-2}\left\{s(s-1)(s-2)\alpha\frac{n^{s-1}}{(n+\beta)^s}\frac{(n+s-3)!(n-s+1)!}{n!(n-1)!}\right.$$
$$\left.+\frac{s(s-1)}{2}\alpha^2\frac{n^{s-2}}{(n+\beta)^s}\frac{(n+s-3)!(n-s+2)!}{n!(n-1)!}\right\}+O(n^{-m}).$$

Proof. Using binomial theorem, the relation between operators (1.2) and (1.3) can be defined as

$$D_{n,\alpha,\beta}(t^s,x)=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^s dt+(1+x)^{-n}\left(\frac{\alpha}{n+\beta}\right)^s$$
$$=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)\sum_{j=0}^{\infty}\binom{s}{j}\frac{(nt)^j\alpha^{s-j}}{(n+\beta)^s}dt+(1+x)^{-n}\left(\frac{\alpha}{n+\beta}\right)^s$$
$$=\sum_{j=0}^{\infty}\binom{s}{j}\frac{n^j\alpha^{s-j}}{(n+\beta)^s}\left[D_n(t^j,x)-(1+x)^{-n}0^j\right]+(1+x)^{-n}\left(\frac{\alpha}{n+\beta}\right)^s.$$

Using (2.2), we get Lemma (2).

Lemma[6,7] *3.* For $m\in\mathbf{N}\cup\{0\}$, *if*

$$U_{n,m}(x)=\sum_{k=0}^{\infty}p_{n,k}(x)\left(\frac{k}{n}-x\right)^m,$$

then $U_{n,0}(x)=1$, $U_{n,1}(x)=0$ and we have the recurrence relation:

$nU_{n,m+1}(x)=x(1+x),\ [U'_{n,m}(x)+mU_{n,m-1}(x)]$.

Consequently, $U_{n,m}(x)=O(n^{-[(m+1/2)]})$, where $[m]$ is integral part of m.

Lemma[9] *4.* For $m\in\mathbf{N}\cup\{0\}$, if

$\mu_{n,m}(x)=D_{n,\alpha,\beta}((t-x)^m,x)$

$$= \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt + p_{n,0}(x)\left(\frac{\alpha}{n+\beta}-x\right)^m$$

then

$$\mu_{n,0}(x)=1,\ \mu_{n,1}(x)\frac{\alpha-\beta x}{n+\beta}$$

and for $n > m$ we have recurrence relation:

$$(n-m)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)$$

$$+\left\{(m+n,x)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n+2m)\right\}\mu_{n,m}(x)$$

$$-\left(\frac{\alpha}{n+\beta}-x\right)\left\{\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)-1\right\}m\mu_{n,m-1}(x).$$

From the recurrence relation, it is easily verified that for all x $\in [0,\infty)$ we have

$$\mu_{n,m}(x)=O(n^{-[(m+1)/2]}).$$

Lemma[6,7] *5.* There exist the polynomials $q_{i,js}(x)$, on $[0,\infty)$ independent of and such that

$$x^s(1+x)^s\frac{d^s}{dx^s}p_{n,k}(x)=\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(k-nx)^j q_{i,j,s}(x)p_{n,k}(x).$$

3. Main result :

In this section, first we show that the derivative $D^{(s)}_{n,\alpha,\beta}(f(t),x)$ is an approximation process for $f^{(s)}$ $s = 1,2,3.....$ Next, we prove a Voronovskaja type asymptotic formula.

Theorem 1. If $s \in \mathbf{N}$, $f \in C_v[0,\infty)$ for some $v > 0$ and $f^{(s)}$ exists at a point x $\in(0,\infty)$ then

$$\lim_{n\to\infty} D^{(s)}_{n,\alpha,\beta}(f,x)=f^{(s)}(x) \qquad (3.1)$$

Proof. By the hypothesis, we have

$$f(t)=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^s,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$. Hence

$$D^{(s)}_{n,\alpha,\beta}(f(t),x)=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}D^{(s)}_{n,\alpha,\beta}((t-x)^i,x)$$

$$+D^{(s)}_{n,\alpha,\beta}\left(\varepsilon(t,x)(t-x)^s,x\right)=I_1+I_2,(\text{say}).$$

Applying Lemma 2, and we have

$$I_1=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}D_{n,\alpha,\beta}\left(t^j,x\right)$$

$$=\frac{f^{(s)}(x)}{s!}\frac{d^s}{dx^s}\left[x^s\frac{n^s}{(n+\beta)^s}\frac{(n+s-1)!\,(n-s)!}{n!\,(n-1)!}+\text{terms in lower power of } x\right]$$

$$=f^{(s)}(x)\frac{n^s}{(n+\beta)^s}\frac{(n+s-1)!\,(n-s)!}{n!\,(n-1)!}$$

$$\to f^{(s)}(x),\text{ as } n\to\infty.$$

Next, using Lemma 5, we obtain

$$I_2=\sum_{\substack{2i+j\le s\\ i,\,j\ge 0}}\frac{n^i q_{i,j,s}(x)}{x^s(1+x)^s}\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^j\int_0^{\infty}b_{n,k}(t)\varepsilon(t,x)\left(\frac{nt+\alpha}{n+\beta}-x\right)^s dt$$

$$+(-1)^s\frac{(n+s-1)!}{(n-1)!}(1+x)^{-n-s}\varepsilon\left(\frac{\alpha}{n+\beta},x\right)\left(\frac{\alpha}{n+\beta}-x\right)^s.$$

Hence

$$|I_2|\le\sum_{\substack{2i+j\le s\\ i,\,j\ge 0}}\frac{n^i\left|q_{i,j,s}(x)\right|}{x^s(1+x)^s}\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^{\infty}b_{n,k}(t)|\varepsilon(t,x)|\left|\frac{nt+\alpha}{n+\beta}-x\right|^s dt$$

$$+\frac{(n+s-1)!}{(n-1)!}(1+x)^{-n-s}\left|\varepsilon\left(\frac{\alpha}{n+\beta},x\right)\right|\left|\frac{\alpha}{n+\beta}-x\right|^s=I_3+I_4.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x,$ for a given $\varepsilon>0$ there exists a $\delta>0$ such that $\varepsilon(t,x)<\varepsilon,$ whenever $0<|t-x|<\delta.$ Further, if γ is any integer $\ge\max(\nu,s),$ then we can find a constant $K>0$ such that $|\varepsilon(t,x)|\left(\frac{nt+\alpha}{n+\beta}-x\right)^s\le K\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\gamma}$ for $|t-x|\ge\delta.$ Hence

$$|I_3|\le C_1\sum_{\substack{2i+j\le s\\ i,\,j\ge 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\left\{\varepsilon\int_{|t-x|<\delta}b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^s dt\right.$$

$$+\int\limits_{|t-x|\,\geq\,\delta} b_{n,k}(t)\,K\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\gamma} dt\Bigg\} = I_5 + I_6, (\text{say}),$$

where $C_1 = \sup\limits_{\substack{2i+j\,\leq\,s\\ i,\,j\,\geq\,0}} \dfrac{|q_{i,j,s}(x)|}{x^s(1+x)^s}$.

Now, on application of Schwarz inequality for integration and then for summation, we obtain

$$|I_5| \leq C_1\,\varepsilon \sum_{\substack{2i+j\,\leq\,s\\ i,\,j\,\geq\,0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \left(\int_0^{\infty} b_{n,k}(t)\,dt\right)^{1/2} \left(\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2s} dt\right)^{1/2}$$

$$\leq C_1\,\varepsilon \sum_{\substack{2i+j\,\leq\,s\\ i,\,j\,\geq\,0}} n^i \left(\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right)^{1/2} \left(\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2s} dt\right)^{1/2}.$$

UsingLemma 3, we get

$$\begin{aligned}\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j} &= n^{2j}\left\{\sum_{k=0}^{\infty} p_{n,k}(x)\,(k/n-x)^{2j}) - (1+x)^{-n}(-x)^{2j}\right\}\\ &= n^{2j}\left\{O\left(n^{-j}\right)+O\left(n^{-r}\right)\right\} \quad (\text{for any}\, r>0)\\ &= O\left(n^{j}\right). \qquad (3.2)\end{aligned}$$

Similarly, using Lemma 4, we get

$$\begin{aligned}\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2s} dt &= O\left(n^{s}\right)-(1+x)^{-n}(-x)^{2s}\\ &= O\left(n^{-s}\right)+O\left(n^{-r}\right) \qquad (\text{for any}\, r>0)\\ &= O\left(n^{-s}\right). \qquad (3.3)\end{aligned}$$

Hence

$$|I_5| \leq C_1\,\varepsilon \sum_{\substack{2i+j\,\leq\,s\\ i,\,j\,\geq\,0}} n^i\, O\left(n^{j/2}\right) O\left(n^{-s/2}\right) = \varepsilon\, O(1).$$

Again, using Schwarz inequality for integration and then for summation, in view of (3.2) and (3.3), we have

$$\left|I_6\right| \leq C_2 \sum_{\substack{2i+j\leq s\\ i,\,j\geq 0}} n^i \sum_{k=1}^{\infty} p_{n,k}(x)\left|k-nx\right|^j \int_{|t-x|\geq\delta} b_{n,k}(t) K \left|\frac{nt+\alpha}{n+\beta}-x\right|^{\gamma} dt$$

$$\leq C_2 \sum_{\substack{2i+j\leq s\\ i,\,j\geq 0}} n^i \left(\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2\gamma} dt\right)^{1/2}$$

$$= \sum_{\substack{2i+j\leq s\\ i,\,j\geq 0}} n^i O\left(n^{j/2}\right)O\left(n^{-\gamma/2}\right)=o(1).$$

and therefore, in view of the arbitrariness of $\varepsilon > 0$, it follows that $I_3 = o(1)$ Also, $I_4 \to 0$, as $n\to\infty$ and hence $I_2 = o(1)$ Combining the estimates of I_1 and I_2 , we obtain (3.1). Next, we prove a Voronovskaja type asymptotic formula.

Theorem 2. Let $f \in C_v[0, \infty)$ for some $v > 0$, and $f^{(s+2)}$ exists at a point $x\in(0, \infty)$ then

$$\lim_{n\to\infty} n\left(D^{(s)}_{n,\alpha,\beta}(f, x) - f^{(s)}(x)\right) = s(s-1-\beta) f^{(s)}(x)$$
$$+\left[(2s-\beta)x+(s+\alpha)\right] f^{(s+1)}(x) + x(1+x) f^{(s+2)}(x). \tag{3.4}$$

Proof. By the Taylor's expansion, we have

$$f(t) = \sum_{i=0}^{s+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{s+2},$$

where $\varepsilon(t,x) \to 0$ as $t\to x$ and $\varepsilon(t,x) = O\left((t-x)^{\delta}\right)$ as $t\to\infty$ for some $\delta > 0$. Thus, we can write

$$n\left\{D^{(s)}_{n,\alpha,\beta}(f(t), x) - f^{(s)}(x)\right\} = n\left\{\sum_{i=0}^{s+2} \frac{f^{(i)}(x)}{i!} D^{(s)}_{n,\alpha,\beta}\left((t-x)^i, x\right) - f^{(s)}(x) + D^{(s)}_{n,\alpha,\beta}\left(\varepsilon(t,x)(t-x)^{s+2}, x\right)\right\}$$

$$= L_1 + L_2, \text{ (say)}.$$

Applying Lemma 2, we have

$$L_1 = n\sum_{i=0}^{s+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} D_{n,\alpha,\beta}\left(t^j, x\right) - f^{(s)}(x)$$

$$= \frac{f^{(s)}(x)}{s!} n\left(D_{n,\alpha,\beta}\left(t^s, x\right) - s!\right) + \frac{f^{(s+1)}(x)}{(s+1)!} n\left\{(s+1)(-x) D_{n,\alpha,\beta}\left(t^s, x\right) + D_{n,\alpha,\beta}\left(t^{s+1}, x\right)\right\}$$

$$+\frac{f^{(s+2)}(x)}{(s+2)!}\, n\left\{\frac{(s+2)(s+1)}{2}\, x^2 D_{n,\alpha,\beta}\left(t^s,x\right)+(s+2)\,(-x)D_{n,\alpha,\beta}\left(t^{s+1},x\right)+D_{n,\alpha,\beta}\left(t^{s+2},x\right)\right\}$$

$$=\frac{f^{(s)}(x)}{s!}\, n\left\{\frac{s!\, n^s (n+s-1)!\,(n-s)!}{(n+\beta)^s\, n!(n-1)!}-s!\right\}+\frac{f^{(s+1)}(x)}{(s+1)!}\, n\Big\{(s+1)\,(-x)\frac{s!\, n^s (n+s-1)!\,(n-s)!}{(n+\beta)^s\, n!(n-1)!}$$

$$+\frac{(s+1)!\, n^{s+1}(n+s)!\,(n-s-1)!}{(n+\beta)^{s+1}\, n!(n-1)!}\, x+\frac{s!(s+1)\, s!\, n^{s+1}(n+s-1)!\,(n-s-1)!}{(n+\beta)^{s+1}\, n!(n-1)!}$$

$$+\frac{s!(s+1)\alpha n^s(n+s-1)!(n-s)!}{(n+\beta)^{s+1} n!(n-1)!}\Big\}$$

$$+\frac{f^{(s+2)}(x)}{(s+2)!}\, n\Big\{\frac{(s+2)(s+1)}{2}\, x^2\, \frac{s!\, n^s (n+s-1)!\,(n-s)!}{(n+\beta)^s\, n!(n-1)!}$$

$$-(s+2)x\Big\{(s+1)!\, x\, \frac{n^{s+1}(n+s)!(n-s-1)!}{(n+\beta)^{s+1} n!(n-1)!}+\frac{s!(s+1) s n^{s+1}(n+s-1)!(n-s-1)!}{(n+\beta)^{s+1} n!(n-1)!}$$

$$+\frac{s!(s+1)\alpha n^s(n+s-1)!(n-s)!}{(n+\beta)^{s+1} n!(n-1)!}\Big\}+(s+2)!\, x^2\, \frac{n^{s+2}(n+s+1)!(n-s-2)!}{2(n+\beta)^{s+2} n!(n-1)!}$$

$$+\frac{(s+1)!(s+2)(s+1)\, n^{s+2}(n+s)!\,(n-s-2)!}{(n+\beta)^{s+2}\, n!(n-1)!}\, x+\frac{(s+1)!(s+2)\, x\alpha\, n^{s+1}(n+s)!\,(n-s-1)!}{(n+\beta)^{s+2}\, n!(n-1)!}$$

$$+\frac{s!s(s+2)(s+1)\alpha\, n^{s+1}(n+s-1)!\,(n-s-1)!}{(n+\beta)^{s+2}\, n!(n-1)!}+\frac{s!(s+1)(s+2)\, n^s\,(n+s-1)!\,(n-s)!}{2(n+\beta)^{s+2}\, n!(n-1)!}\Big\}+O\left(n^{-m}\right).$$

The coefficients of $f^{(s)}(x)$, $f^{(s+1)}(x)$, $f^{(s+2)}(x)$ in the above expression are respectively $s(s\text{-}1\text{-}\beta)$, $[(2s-\beta)x+(s+\alpha)]$, $x(1+x)$, which follows by using induction hypothesis on s and taking limits as $n\to\infty$. Hence, to prove (3.4), it is sufficient to show that for each $x\in(0,\infty)$, $n\, L_2\to 0$ as $n\to\infty$, which follows on proceeding along the lines of the proof of $I_2\to 0$ as $n\to\infty$ in Theorem 1.

Remark 1. In particular if s = 0, we obtain the following conclusion of the above asymptotic formula in ordinary approximation:

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