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S_1-Near subtraction semigroups

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Abstract

In this paper we introduce the concepts of S_1-near subtraction semigroups, that is near subtraction semigroups X satisfying the condition: for every $a \in X$ there exists $x \in X - \{0\}$ such that $axa = xa$. We also discuss some of their properties and obtain certain characterisation theorems.

Key Words: S_1-near subtraction semigroup, nil near subtraction semigroup, mate function, nilpotents, idempotents.

Mathematics Subject Classification: 06F35.

Introduction

A non empty set X together with a binary operation '$-$' is said to be a subtraction algebra if it satisfies the following axioms:

(i) $x-(y-x)=x$

(ii) $x-(x-y)=y-(y-x)$

(iii) $(x-y)-z=(x-z)-y$ for every $x, y, z \in X$.

A non empty set X together with two binary operations '$-$' and '$\cdot$' is said to be a right near subtraction semigroup if it satisfies the following:

(i) $(X, -)$ is a subtraction algebra.

(ii) $(X, \cdot)$ is a semigroup.

(iii) $(x-y)\cdot z = x\cdot z - y\cdot z$ for all $x, y, z \in X$.

We shall henceforth write xy for $x\cdot y$ for any two elements x, y of X.

Throughout this paper, X stands for a (right) near subtraction semigroup $(X, -, \cdot)$ with at least two elements. The subtraction

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determines an order relation on $X : a \leq b \Leftrightarrow a - b = 0$ where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. In X, $0 - x = 0$ and $0x = 0$ for all $x \in X$. If $x0 = 0$ for all $x \in X$, we say X is zero symmetric.

We recall the following definitions from[6]: A near subtraction semigroup X is said to have (i) IFP (Insertion of Factors Property) if for a, b in X, $ab = 0 \Rightarrow axb = 0$ for all $x \in X$ (ii) (*, IFP) if X has IFP and $ab = 0 \Rightarrow ba = 0$ for $a, b \in X$ (iii) Strong IFP if for all ideals I of X, $ab \in I \Rightarrow axb \in I$ for all x in X. If $A, B \subset X$, then $AB = \{ab / a \in A, b \in B\}$.

We say that subset Y of X which is closed under '$-$' and $XY \subset Y$ is called an X-system and if in addition, $YX \subset Y$ then Y is called an invariant X-system[1-3].

If there exists a map $f : X \to X$ such that $a = a\,f(a)\,a$ for all a in X then f is called a mate function for X. We say that X is an $S(S')$ near subtraction semigroup if $a \in Xa\,(aX)$ for all $a \in X$.

2. *Notation :*

a) An element $e \in X$ is said to be (i) idempotent if $e^2 = e$ (ii) nilpotent if $e^k = 0$ for some positive integer k (X is said to be nil if every element of X is nilpotent) (iii) right identity if $ea = a$ for every $a \in X$.

b) If $a^2 = 0 \Rightarrow a = 0$ for all $a \in X$, then X has no non zero nilpotent elements (as in problem 14, p.9 of [4]).

c) $X_d = \{n \in X \mid n(x - y) = nx - ny$, for all $x, y \in X\}$ -the set of all distributive elements of X.

d) The centre of X is defined as $C(X) = \{a \in X / ax = xa$ for all $x \in X\}$.

e) E denotes the set of all idempotents of X.

f) L denotes the set of all nilpotent elements of X.

g) $X^* = X - \{0\}$.

For definitions and notations used but left undefined in this paper we refer to Pilz[5].

3. S_1-*Near Subtraction Semigroup :*

Let us begin with the following definition:

Definition 3.1: We say that X is an S_1-near subtraction semigroup if for every $a \in X$ there exists $x \in X^*$ such that $axa = xa$.

Examples 3.2 (a) Let $X = \{0, a, b, 1\}$ in which '$-$' and '$.$' are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

This is an S_1-near subtraction semigroup.

(b) We consider the near subtraction semigroup (X,-,.)where $X = \{0, a, b, 1\}$, '$-$' and '$\cdot$' are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

This is not an S_1-near subtraction semigroup.

We shall now prove a characterisation of S_1-near subtraction semigroups.

Theorem 3.3 : Let X be a nil near subtraction semigroup. Then X is an S_1-near subtraction semigroup if and only if X is zero symmetric.

Proof: For the only if part, we take $a \in X$. Since X is an S_1-near subtraction semigroup there exists $x \in X^*$ such that,

$$axa = xa \quad (1)$$

We shall prove that

$$ax^k a = x^k a \quad (2)$$

for all positive integers k. Equation (1) demands that (2) is true for k=1. We assume that the result is true for $k = s - 1$. If $k = s$ then $ax^s a = ax^{s-1}(xa) = ax^{s-1}axa$ (by [1]) $= (ax^{s-1}a)xa = (x^{s-1}a)xa = x^{s-1}(axa) = x^{s-1}(xa) = x^s a$. Thus by induction $ax^k a = x^k a$ for all positive integers k. Since X is nil, $x^t = 0$ for some positive integer t and since $ax^t a = x^t a$ we get $a0a = 0a$. It follows that X is zero symmetric.

For the if part, let $a \in X$. Since X is nil, there exists a positive integer $k > 1$ such that $a^k = 0$. This implies $xa = 0$ where $x = a^{k-1}$. Therefore $axa = a(xa) = a0 = 0$ (since X is zero-symmetric) $= xa$. Thus X is an S_1 near subtraction semigroup.

Proposition 3.4 : Let X be a zero symmetric S_1-near subtraction semigroup. If X has no non-zero zero-divisors then every X-system and every ideal of X is an S_1-near subtraction semigroup in its own right.

Proof: Let A be an X-system of X and $A^* = A - \{0\}$. Let $a \in A^*$. Since X is an S_1-near subtraction semigroup there exists $x \in X^*$ such that $axa = xa$. We take $n = xa \in XA$. Since A is an X-system of X, $n \in A$. Since X has no non-zero zero-divisors, $n \neq 0$. Now $ana = a(xa)a = (axa)a = (xa)a = na$. If $a = 0$ then, since X is zero symmetric, $ana = na$ for any $n \in A^*$. Thus A is an S_1-near subtraction semigroup.

Next, let I be an ideal of X and let $a \in I$. If $a = 0$ then $ana = na$ for any $n \in I^*$. Suppose $a \neq 0$. Since X is an S_1-near subtraction semigroup, there exists $x \in X^*$ such that $axa = xa$. If $i = ax \in IX$, since I is an ideal of X we get $i \in I$.

Our hypothesis demands that $i \neq 0$. Now $aia = a(ax)a = a(axa) = a(xa) = (ax)a = ia$. Thus I is an S_1-near subtraction semigroup.

We furnish below a necessary and sufficient condition for an S_1-near subtraction semigroup to have a mate function.

Proposition 3.5: Let X be an S_1-near subtraction semigroup without non-zero zero-divisors. Then X has a mate function if and only if X is Boolean.

Proof: Let $a \in X$. Since X is an S_1-near subtraction semigroup there exists $x \in X^*$ such that $axa = xa$.

For the 'only if ' part, we assume that, X has a mate function 'f'. Then $a = af(a)a$. We have, $af(a)xa = af(a)(xa) = af(a)(axa) = af(a)a(xa) = axa$. That is, $af(a)xa = axa$ and this implies $af(a)xa - axa = 0 \Rightarrow (af(a) - a)xa = 0$. Since X has no non-zero zero-divisors, $af(a) - a = 0$. That is $af(a) = a$. Therefore $(af(a))a = a^2$. This implies that $a = a^2$. Consequently X is Boolean.

Proof of 'if ' part is obvious.

4. S_1-near subtraction semigroup and the subsets:

In this section we introduce the notation $X_{S_1}(a)$ [8], $a \in X$ and discuss the properties of S_1-near subtraction semigroup with the subset $X_{S_1}(a)$.

Notation 4.1: For any $a \in X$, we denote $\{x \in X^* / axa = xa\}$ by $X_{S_1}(a)$.

Proposition 4.2: It easily follows that X is an S_1-near subtraction semigroup if and only if $X_{S_1}(a) \neq \phi$ for all $a \in X$.

Proof: Straight forward.

Example 4.3 (a) We consider the S_1-near subtraction semigroup X cited in Example 3.2(a). We observe that $X_{S_1}(a) \neq \phi$ for all

$a \in X$.

(b) In the case of Example 3.2(b), $X_{S_1}(b) = \phi$.

Proposition 4.4: Let X be an S_1-near subtraction semigroup. If $ab=ba$ for $a, b \in X$ and if $X_{S_1}(a) \cap X_{S_1}(b) \neq \phi$ then $X_{S_1}(a) \cap X_{S_1}(b) \subset X_{S_1}(ab)$.

Proof: Let $a, b \in X$. Suppose $ab = ba$. Let $x \in X_{S_1}(a) \cap X_{S_1}(b)$ then $x \in X_{S_1}(a)$ and $x \in X_{S_1}(b)$ *i.e.*, $axa = xa$ and $bxb = xb$. Now $(ab)x(ab) = (ba)x(ab) = b(axa)b = b(xa)b = bx(ab) = bx(ba) = (bxb)a = (xb)a = x(ba) = x(ab)$.

It follows that $x \in X_{S_1}(ab)$ and hence $X_{S_1}(a) \cap X_{S_1}(b) \subset X_{S_1}(ab)$.

Lemma 4.5: Let X be an S_1-near subtraction semigroup. Then $X_{S_1}(a)$ has no non-zero zero-divisors if and only if $X_{S_1}(a)$ is a multiplicative system.

Proof: Since X is an S_1-near subtraction semigroup, $X_{S_1}(a) \neq \phi$ for all $a \in X$ (by Propsoition 4.2). For the 'only if ' part, let $x, y \in X_{S_1}(a)$. Then $x, y \in X^*$ and $axa = xa$, $aya = ya$. It follows that $a(xy)a = ax(ya) = ax(aya) = (axa)ya = (xa)ya = x(aya) = x(ya) = (xy)a$. That is $a(xy)a = (xy)a$. Further since $X_{S_1}(a)$ has no non-zero zero divisors, $xy \neq 0$. Consequently, $xy \in X_{S_1}(a)$ and $X_{S_1}(a)$ is a multiplicative system.

For the 'if ' part, let $x, y \in X_{S_1}(a)$. Since $X_{S_1}(a)$ is a multiplicative system $xy \in X_{S_1}(a)$. As $X_{S_1}(a) \subset X^*$, it follows that $xy \neq 0$ and hence $X_{S_1}(a)$ has no non-zero zero-divisors.

We conclude our discussion with theorems 4.7 where we prove some important properties of an S_1-near subtraction semigroup. Before that we have:

Lemma 4.6 : If X is an S_1-near subtraction semigroup then $X_{S_1}(a) \subset X_{S_1}(a^k)$ for all positive integers $k > 1$ and for all $a \in X$.

Proof: Let $x \in X_{S_1}(a) \Rightarrow axa = xa$. Therefore $a^2xa^2 = a(axa)a = a(xa)x = (axa)a = (xa)a = xa^2$. That is $a^2xa^2 = xa^2$. Continuing in the same vein we get, $a^kxa^k = xa^k$ for all positive integer k. It follows that $X_{S_1}(a) \subset X_{S_1}(a^k)$.

Theorem 4.7: Let X be an S_1-near subtraction semigroup, without non-zero zero divisors. Then we have the following:

(i) $aX_{S_1}(a) = X_{S_1}(a)$ for all $a \in X^*$.

(ii) $X_{S_1}(a)\, a \subset X_{S_1}(a)$ for all $a \in X^*$.

(iii) If X is finite then $X_{S_1}(a)a = X_{S_1}(a)$ for all $a \in X^*$.

(iv) If X is zero symmetric then $X_{S_1}(a)X^* \subset X_{S_1}(a)$ for all $a \in X$.

(v) $[X_{S_1}(a)]^k \subset X_{S_1}(a^k)$ for all positive integers $k > 1$ and for all $a \in X$.

Proof: (i) Let $z \in aX_{S_1}(a)$. Then there exists $x \in X_{S_1}(a)$ such that $z = ax$. Since $x \in X_{S_1}(a)$, $axa = xa$. Now $aza = a(ax)a = a(axa) = a(xa) = (ax)a = za$. Since X has no non-zero zero-divisors, $z \in X^*$. It follows that $z \in X_{S_1}(a)$ and therefore[7]

$$aX_{S_1}(a) \subset X_{S_1}(a) \tag{3}$$

Let $y \in X_{S_1}(a)$. Then $aya = ya$. That is $(ay - y)a = 0$. Since X has no non-zero zero-divisors, $ay - y = 0$. This implies $y = ay \in aX_{S_1}(a)$ and therefore

$$X_{S_1}(a) \subset aX_{S_1}(a) \tag{4}$$

From (3) and (4) we get $aX_{S_1}(a) = X_{S_1}(a)$.

(ii) If $z \in X_{S_1}(a)$ then $aza = za$. Clearly then $a(za)a = (aza)a = (za)a$. That is $a(za)a = (za)a$. Since X has no non-zero divisors $za \neq 0$. Consequently, $za \in X_{S_1}(a) \Rightarrow X_{S_1}(a)a \subset X_{S_1}(a)$.

(iii) Suppose X is a finite S_1-near subtraction semigroup. Let $x_1, x_2, \dots, x_s$ be all the elements of $X_{S_1}(a)$. From (ii) we get $x_1a, x_2a, \dots, x_sa \in X_{S_1}(a)$ for all $a \in X^*$. We prove that they all are distinct. Suppose $x_ka = x_la$ for $k \neq l$. Then $(x_k - x_l)a = 0$. Since X has no non-zero divisors we get $x_k - x_l = 0$. That is $x_k = x_l$ which is a contradiction to $k \neq l$. Therefore $x_1a, x_2a, \dots, x_sa$ are s-distinct elements lying in $X_{S_1}(a)$. Thus every element $y \in X_{S_1}(a)$ can be written as $y = x_ta$ for some x_t in $X_{S_1}(a)$. Therefore $X_{S_1}(a) \subset X_{S_1}(a)a$ and by using (ii) we get $X_{S_1}(a)a = X_{S_1}(a)$.

(iv) We observe that X is zero-symmetric since it is an S_1-near subtraction semigroup by Theorem

3.3. For any $a, n \in X^*$, $x \in X_{S_1}(a) \Rightarrow axa = xa \Rightarrow (ax - x)a = 0$. Since X has no non-zero zero-divisors, $ax - x = 0$. Therefore $(ax - x)na = 0na = 0$. This implies that $axna - xna = 0$. That is $a(xn)a = (xn)a$. Since X has no non-zero zero-divisors, $xn \neq 0$. Therefore $xn \in X_{S_1}(a)$ and this implies $X_{S_1}(a)X^* \subset X_{S_1}(a)$. If $a = 0$, then $0(xn)0 = (xn)0$ (since X is zero-symmetric. Consequently $X_{S_1}(a)X^* \subset X_{S_1}(a)$ for all $a \in X$.

(v) Since X has non-zero zero divisors, $X_{S_1}(a)$ has no non-zero zero divisors for all $a \in X$. Now Lemma 4.5 demands that $X_{S_1}(a)$ is a multiplicative system. That is $xy \in X_{S_1}(a)$ for all $x, y \in X_{S_1}(a)$. Therefore $[X_{S_1}(a)]^2 \subset X_{S_1}(a)$. Proceeding this way,

$$[X_{S_1}(a)]^k \subset X_{S_1}(a) \tag{5}$$

for all positive integers $k > 1$. Appealing to Lemma 4.6 we get.

$$X_{S_1}(a) \subset X_{S_1}(a^k) \tag{6}$$

Comibining (5) and (6) we get $[X_{S_1}(a)]^k \subset X_{S_1}(a^k)$.

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