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On $|N,p_n;\delta|_k$ Summability of orthogonal series

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Abstract

In this paper we have proved a theorem on $|N,p_n;\delta|_k$ summability of orthogonal series, which generalizes various known results. However, our theorem is as follows:-

Theorem : Let $1 \le k \le 2$, and $0\le\delta_k<1$. If $\{p_n\}$ be a positive non-increasing sequence such that

$P_n \to\infty$ as $n \to\infty$ and

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right\}$$

also $\{\lambda_n\}$ is positive sequence and the series

$$\sum_{n=1}^{\infty}\left(\frac{p_m}{p_n}\right)^{\delta k-1}\frac{1}{p_{n-1}^{k}}\left\{\sum_{j=1}^{n}p_{n-j}^{2}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^{2}\lambda_j^2\left|a_j^2\right|\right\}^{k/2}$$

converges, and if

$$\sum_{v=1}^{\infty}\left(\frac{p_v}{P_v}\right)\left|\left(\frac{P_v}{p_v}\right)\Delta\lambda_v+\lambda_{v+1}\right|^{k}<\infty$$

then the orthogonal series $\Sigma\, \lambda_n a_n \varnothing_n(x)$ is summable $|N,p_n;\delta|_k$ almost everywhere[1-4].

Key words and phrases : $|N,p_n;\delta|_k$ summability, orthogonal series.

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Introduction

1. *Definitions and Notations :*

Let Σa_n be given infinite series with s_n as its n-th partial sum. If p_n is a sequence of positive numbers such that[5-8]

$$P_n = \sum_{v=0}^{n} p_v \to \infty$$

$$\text{as } n \to \infty$$

$$(P_{-i} = p_{-i} = 0,\ i \ge 1)$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}\, s_v$$

$$= \frac{1}{P_n}\sum_{v=0}^{n} p_v a_{n-v},\quad P_n \neq 0 \tag{1.1}$$

defines the sequence T_n of the (N, p_n) means of the sequence s_n generated by the sequence of coefficients p_n.

The series Σa_n is said to be summable $|N, p_n|_k$, $k \ge 1$ if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty \tag{1.2}$$

The case k=1 is reduced to the Nörlund summability $|N, p_n|$ and further, in the special case in which

$$p_n = \frac{A_n^{\delta-1}}{} = \binom{n+\delta-1}{n} \text{ and } p_n = \frac{1}{(n+1)},$$ the

summability $|N, p_n|$ is the same as the summability $|C, \delta|$ and the absolute harmonic summability, respectively.

The series Σa_n is said to be summable $|N, p_n, \delta|_k$, $k \ge 1$, $\delta \ge 0$, if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_n - T_{n-1}|^k < \infty$$

Let $\{\varnothing_n(x)\}$ be an orthogonal system defined in the interval (a,b). For a function $f(x) \in L^2$ (a,b) such that

$$f(x) \approx \sum_{n=0}^{\infty} a_n \varnothing_n(x) \tag{1.3}$$

We denote by $E_n^{(2)}(f)$ the best approximation to f(x) in the metric L^2 by means of polynomials of $\varnothing_0(x), \ldots\ldots \varnothing_{n-1}(x)$. It is well known that

$$E_n^{(2)}(f) = \left(\sum_{j=n}^{\infty}|a_j|^2\right)^{1/2}$$

We put $\Delta p_n = \lambda_n - \lambda_{n-1}$ for any sequence $\{\lambda_n\}$ A is a positive constant necessarily the same at each occurrence[1-7].

2. Dealing with the absolute Nörlund summability of orthogonal series, Okuyama[7] has proved the following theorem.

Theorem A : Let $1 \le k \le 2$ and $\{\lambda_n\}$ be a positive sequence. If $\{p_n\}$ is a positive sequence and the series

$$\sum_{n=1}^{\infty}\frac{p_{n^k}}{p_n\, p_{n-1}}\left\{\sum_{j=1}^{n} p_{n-j}^2\left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right)^2 \lambda_j^2 \left|a_j^2\right|\right\}^{1/2}$$

converges, the orthogonal series

$$\Sigma\, \lambda_n\, a_n\, \varnothing_n(x) \tag{2.1}$$

is summable $|N, p_n|_k$ almost everywhere[7-10].

In 1998 Sönmez[8] has extended the

theorem of Okuyama to $|N, p_n, \delta|_k$ summability of orthogonal series in the following form.

Theorem B : Let $1\le k\le 2$ and $0\le\delta_k<1$. If $\{p_n\}$ and $\{\lambda_n\}$ are positive sequences and the series

$$\sum_{n=1}^{\infty}\left(\frac{p_m}{p_n}\right)^{\delta k-1}\frac{1}{p_{n-1}^{k}}\left\{\sum_{j=1}^{n}p_{n-j}^{2}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^{2}\lambda_j^2|a_j^2|\right\}^{k/2}$$

converges, then the orthogonal series $\Sigma\ \lambda_n a_n Ø_n(x)$ is summable $|N, p_n, \delta|_k$ almost everywhere.

The aim of this paper is to generalize Theorem B for $|N, p_n, \delta|_k$ summability.

3. We shall prove the following theorem.

Theorem : Let $1\le k\le 2$, and $0\le\delta_k<1$. If $\{p_n\}$ be a positive non-increasing sequence such that

$Pn\to\infty$ as $n\to\infty$ and

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right\} \quad (3.1)$$

also $\{\lambda_n\}$ is positive sequence and the series

$$\sum_{n=1}^{\infty}\left(\frac{p_m}{p_n}\right)^{\delta k-1}\frac{1}{p_{n-1}^{k}}\left\{\sum_{j=1}^{n}p_{n-j}^{2}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^{2}\lambda_j^2|a_j^2|\right\}^{k/2} \quad (3.2)$$

converges, and if

$$\sum_{v=1}^{\infty}\left(\frac{p_v}{P_v}\right)\left|\left(\frac{P_v}{p_v}\right)\Delta\lambda_v+\lambda_{v+1}\right|^{k}<\infty \quad (3.3)$$

then the orthogonal series $\Sigma\ \lambda_n a_n Ø_n(x)$ is summable $|N, p_n;\ \delta|_k$ almost everywhere. (3.4)

4. *Proof of the theorem :* Let $T_n(x)$ be the n-th Nörlund mean of the series (3.4).

Then we have by (1.1)

$$\Delta T_n(x) = T_n(x) - T_{n-1}(x) = \frac{p_n}{P_nP_{n-1}}\sum_{j=1}^{n}p_{n-j}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)\lambda_j a_j\varphi_j(x).$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n}j\left(\frac{-P_j}{p_j}\Delta T_{j-1}+\frac{P_{j-2}}{p_{j-1}}\Delta T_{j-2}\right)\lambda_j a_j\varphi_j(x).$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}(-j)\left(\frac{P_j}{p_j}\right)\Delta T_{j-1}\lambda_j a_j\varphi_j(x)-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n-1)p_n}\Delta T_{n-1}$$

$$+\frac{1}{(n+1)}\sum_{j=1}^{n}\frac{jP_{j-2}}{P_{j-1}}\Delta T_{j-2}\lambda_j a_j\varphi_j(x)$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}(-j)\frac{P_j}{p_v}\Delta T_{j-1}\lambda_j a_j\varphi_j(x)$$

$$+\frac{1}{(n+1)}\sum_{j=1}^{n-1}(j+1)\frac{P_{j-1}}{p_j}\Delta T_{j-1}\lambda_j a_j\varphi_j(x)$$

$$-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n+1)p_n}\Delta T_{n-1}$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}\frac{\Delta T_{j-1}}{p_v}\{-j\lambda_j a_j\varphi_j(x)P_j+(j+1)\lambda_{j-1}a_{j-1}\varphi_{j-1}(x)P_{j-1}\}$$

$$-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n+1)p_n}\Delta T_{n-1}$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}\frac{\Delta T_{j-1}}{p_j}\{-j\lambda_j a_j\varphi_j(x)P_j+(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)(P_j-p_j)\}$$

$$-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n+1)p_n}\Delta T_{n-1}$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}\frac{\Delta T_{j-1}}{p_j}\{-j\lambda_j a_j\varphi_j(x)P_j+(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)P_j-(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)p$$

$$\}-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n+1)p_n}\Delta T_{n-1}$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}\frac{\Delta T_{j-1}}{p_j}\{-[\Delta j\lambda_j a_j\varphi_j(x)]P_j-(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)p_j\}$$

$$-\frac{nP_n\lambda_j a_j\varphi_j(x)}{(n+1)p_n}\Delta T_{n-1}$$

$$=\frac{1}{(n+1)}\sum_{j=1}^{n-1}\left(\frac{P_j}{p_j}\right)\Delta T_{j-1}\left[j\Delta\Delta_j a_j\varphi_j(x)-\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right]$$

$$-\frac{1}{(n+1)}\sum_{j=1}^{n-1}\Delta T_{j-1}(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)$$

$$-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n+1)p_n}\Delta T_{n-1}$$

$$=-\frac{1}{(n+1)}\sum_{i=1}^{n-1}\frac{jP_j}{p_i}\Delta\lambda_j a_j\varphi_j(x)\Delta x_{j-1}+\frac{1}{(n+1)}\sum_{i=1}^{n-1}\frac{P_j}{p_i}\Delta T_{j-1}\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)$$

$$-\frac{1}{(n+1)}\sum_{j=1}^{n-1}(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\Delta T_{j-1}$$

$$-\frac{nP_n\lambda_n a_n\varphi_n(x)}{(n+1)p_n}\Delta T_{n-1}$$

$= \tau_{n,1} + \tau_{n,2} + \tau_{n,3} + \tau_{n,4}$ (say).

To prove the theorem it is suffcient to show that

$$\sum_{n=1}^{\infty}|\tau_{n,i}|^k < \infty \qquad \text{for } i=1,2,3,4.$$

$$\sum_{n=1}^{\infty}|\tau_{n,1}|^k = \sum_{n=1}^{\infty}\left|\frac{1}{(n+1)}\sum_{j=1}^{n-1}\frac{jP_j}{p_j}\Delta\lambda_j a_j\varphi_j(x)\Delta T_{j-1}\right|^k$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n^k}\left\{\sum_{j=1}^{n-1}\frac{P_j}{p_j}\, j|\lambda_j a_j\varphi_j(x)||\Delta T_{j-1}|\right\}^k$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k j^k|\Delta\lambda_j a_j\varphi_j(x)|^k$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k j^k|\Delta\lambda_j a_j\varphi_j(x)|^k$$

$$= \sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k j^k|\Delta\lambda_j a_j\varphi_j(x)|^k\sum_{n=j+1}^{\infty}\frac{1}{n}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\frac{1}{j}|\Delta T_{j-1}|^k j^k|\Delta\lambda_j a_j\varphi_j(x)|^k\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\left(\frac{P_j}{jp_j}\right)|\Delta T_{j-1}|^k|j\Delta\lambda_j a_j\varphi_j(x)|^k\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k\right\}$$

$= O(1)$

by condition of the theorem.

Again

$$\sum_{n=1}^{\infty}|\tau_{n,2}|^k = \sum_{n=1}^{\infty}\left|\frac{1}{(n+1)}\sum_{j=1}^{n-1}\frac{P_j}{p_j}\Delta T_{j-1}\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right|^k$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n^k}\left\{\sum_{j=1}^{n-1}|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)|\left|\frac{P_j}{p_j}\right||\Delta T_{j-1}|\right\}^k$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{j=1}^{n-1}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)|^k$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)|^k\sum_{n=j+1}^{\infty}\frac{1}{n}\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\frac{1}{j}|\Delta T_{j-1}|^k|\lambda_{j+1}a_{j+1}\varphi_{j+1}|\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k\frac{P_j}{jP_j}\Big|\lambda_{j+1}a_{j+1}\varphi_{j+1}\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k\frac{P_j(j+1)p_{j+1}}{jp_jP_{j+1}}\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}|\Delta T_{j-1}|^k\right\}$$

$=O(1)$

Similarly

$$\sum_{n=1}^{\infty}|\tau_{n,3}|^k = \sum_{n=1}^{\infty}\left|\frac{1}{(n+1)}\sum_{j=1}^{n-1}(j+1)\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\Delta T_{j-1}\right|^k$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n^k}\left\{\sum_{j=1}^{n-1}(j+1)|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)||\Delta T_{j-1}|\right\}^k$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^k}\left\{\sum_{j=1}^{n-1}\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)|\Delta T_{j-1}| -\right.$$

$$= O\left\{\sum_{n=1}^{\infty}\frac{1}{n^k}\left\{\sum_{j=1}^{n-1}j|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)||\Delta T_{j-1}|\right\}^k\right\}$$

$$= O\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{j=1}^{n-1} j^k\left|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right|^k\left|\Delta T_{j-1}\right|^k\right\}\right\}$$

$$= O\left\{\sum_{n=1}^{\infty} j^k\left|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right|\right\}^k\left|\Delta T_{j-1}\right|^k\sum_{n=j+1}^{\infty}\frac{1}{n}$$

$$= O\left\{\sum_{j=1}^{\infty} j^{k-1}\left|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right|^k\left|\Delta T_{j-1}\right|^k\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\left(\frac{P_j}{j}\right)^{\delta k-1} j^{k-1}\left|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right|^k\left|\Delta T_{j-1}\right|\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\left|\Delta T_{j-1}\right|^k\left(\frac{jp_j}{P_j}\right)^{\delta k}\left(\frac{jp_j}{P_j}\right)^{-1}\left|\lambda_{j+1}a_{j+1}\varphi_{j+1}(x)\right|^k\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\left|T_{j-1}\right|^k\left|\frac{P_j}{jp_j}\frac{(j+1)p_{j+1}}{P_{j+1}}\right|\right\}$$

$$= O\left\{\sum_{j=1}^{\infty}\left(\frac{P_j}{p_j}\right)^{\delta k-1}\left|T_{j-1}\right|^k\right\}$$

$= O(1)$,

Lastly

$$\sum_{n=1}^{\infty}\left|\tau_{n,4}\right|^k = \sum_{n=1}^{\infty}\left|\frac{nP_n}{(n+1)P_n}\lambda_n a_n\varphi_n(x)\Delta T_{n-1}\right|^k$$

$$= O\left\{\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k}\left|\lambda_n a_n\varphi_n(x)\right|^k\left|\Delta T_{n-1}\right|^k\right\}$$

$$= O\left\{\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\left|\Delta T_{n-1}\right|^k\left(\frac{P_n}{np_n}\right)\left|\lambda_n a_n\varphi_n(x)\right|^k\right\}$$

$$= O\left\{\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\left|\Delta T_{n-1}\right|^k\right\}$$

$= O(1)$.

This completes the proof of the theorem[9-10]

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