Ultra Scientist Vol. 24(3)A, 601-603 (2012).

Construction of symmetric weighing matrix of weight 4

NARENDRA PRASAD[1] and DIPSHKHA BHATTACHARYA[2]

(Acceptance Date 16th October, 2012)

Abstract

The aim of this paper is to construct the symmetric weighing matrix W of weight 4 with the help of (0,1,-1) matrix M satisfying $M^2 = 4M$.

Introduction

2. *Notations :*

J_n : nXn matrix having all entries 1

O_n : nXn matrix having all entries 0

AXB : Kronecker Product of two matrices A & B.

Def : Kronecker Product of two matrices A and B is defined as

$$AXB = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{11} & a_{22} & a_{2n} \\ \text{-------} & \text{-------} & \text{-------} \\ \text{-------} & \text{-------} & \text{-------} \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix} XB = \begin{pmatrix} a_{11}B & a_{12}B & a_{1n}B \\ a_{21}B & a_{22}B & a_{2n}B \\ \text{-------} & \text{-------} & \text{-------} \\ \text{-------} & \text{-------} & \text{-------} \\ a_{m1}B & a_{m2}B & a_{mn}B \end{pmatrix}$$

3. *Weighing Matrix :*

Let n and w be positive integers. A weighing matrix W of weight ω and order n is an nXn (0,±1) matrix satisfying $WW^T = \omega I_n$. Such a matrix is denoted by W(n,w) (ω =0 is generally not permiteed)[1-4].

Examples

A is W(4,3) ; B is W (6,5)

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

Weighing matrix is a special type of orthogonal design.

4. *Symmetric Weighing matrix :*

A Symmetric weighing matrix W is a (0,1,-1) matrix of order n if it satisfies.

$W^2 = mI_n$ where m is called the weight of symmetric weighing matrix W.

Example:

The matrix $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Is a symmetric weighing matrix of weight 1.

Theorem:

A Symmetric weighing matrix of weight 4 with diagonal element -1 exists if a (0,1,-1) symmetric matrix M satisfies $M^2 = 4M$.

Proof : We suppose that a (0,1,-1) Symmetric matrix M satisfies $M^2 = 4M$ (1)

Then we have to prove that a symmetric weighing matrix W of weight 4 with diagonal element -1 exist[5-7].

Let $W = M - 2I_n$.

Which is possible due to (1), Where in is identity matrix of order n.

We Consider
$$\begin{aligned} W^2 &= (M-2I_n)^2 \\ &= M^2 - 4MI_n + 4I_n \\ &= 4M - 4M + 4I_n \quad \text{by (1)} \\ \therefore W^2 &= 4I_n \end{aligned}$$

Which shows that W is a symmetric weighing matrix of weight 4.

Conversely suppose that W is a symmetric weighing matrix of weight 4. Then we have to prove that matrix M satisfies $M^2 = 4M$.----(3)

$\because$ Weight of a symmetric weighing matrix W is 4

$$\therefore W^2 = 4I_n$$

We take $W = M - 2I_n$ and

Consider $W^2 = (M-2I_n)^2$

$\Rightarrow 4I_n = M^2 - 4MI_n + 4I_n$

$\Rightarrow M^2 - 4M = 0$

$\Rightarrow M^2 = 4M.$

Example 1 If a (0,1) matrix M of order 4m (m≥2) is in the form as

$$M = \begin{pmatrix} J_2 & O_2 & O_2 \text{-----} & O_2 & J_2 \\ O_2 & J_2 & O_2 \text{-----} & J_2 & O_2 \\ \text{------} & & & & \\ \text{------} & & & & \\ \text{------} & & & & \\ J_2 & O_2 & O_2 \text{-----} & O_2 & J_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \text{ ----} & 0 & 1 \\ 0 & 1 & 0 \text{ ----} & 1 & 0 \\ \text{------} & & & & \\ \text{------} & & & & \\ \text{------} & & & & \\ 1 & 0 & 0 \text{ ----} & 0 & 1 \end{pmatrix} X J_2$$

Satisfying $M_2 = 4M$ then there exists an infinite class of symmetric weighing matrix $W = M - 2I_{4m}$ of weight 4 and of order 4m.

For let m=2 then the matrix M is given as.

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} X\ J_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} X \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$M^2 = 4M$$

The matrix $W = M - 2I_8$ ie

$$W = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Is a 8 X 8 Symmetric weighing matrix of weight 4

$$W^2 = 4I_8$$

Example 2 Let a (0,1,-1) matrix

$$M=\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & +1 \\ -1 & -1 & 1 & -1 \\ 1 & +1 & -1 & 1 \end{pmatrix} \quad X \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Is a 8 X 8 Symmetric weighing matrix of weight 4

ie $W^2 = 4 I_8$

$$M^2 = 4M$$

Then $W = M - 2I_8$ ie

$$W=\begin{pmatrix} -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

Is a 8 X 8 Symmetric weighing matrix of weight 4 ie $W^2 = 4I_8$

References

1. Craigen, R. and Kharghani, H. Orthogonal designs in hand book of Combinatoial designs Second edition edited by J. colbourn and J. H Dinitz chapman and Hall /CR (2007).
2. Chan H.C. Rodger C.A. and saberty, J on equivalent weighing matrices. *Ars Comb. 21*, 299-333 (1996).
3. Craigen, R. A New Class of weighing matrices. with Square weights Bull. Ins / *comb. Appl 3*, 33-42 (1991).
4. Craigen, R., The structure of Weighing matrices having large weights Des. Codes *Crypt 5*, 199-216 (1995).
5. Ohmori, H. Classification of weighing matrices of order 13 and weight 9 *Discrete Math 166*, 55-78 (1993).
6. Kharghani, H - Array for orthogonal designs. *J. Combin Des 8*, 166-173 (2000).
7. Geramita, Anthony, V. Pallman Norman J. and Wallis, Jennifer S Famities of weighing matrices, *Bull Austral Math Soc*, Vol *10*, 19-122 (1974).