ISSN 2231-346X (Print)
JUSPS-A Vol. 28(5), 278-285 (2016). Periodicity-Monthly
(Online) ISSN 2319-8044
Section A
JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

A Mixed Quadrature Rule by Blending Clenshaw-Curtis and Gauss-Legendre Quadrature Rules for Approximate Evaluation of Real Definite Integrals in Two Dimensions

[1] PRITIKANTA PATRA and [2] RAJANI BALLAV DASH*
[1] *E-mail*: [1] pritikanta@yahoo.com, [2] rajani_bdash@gmail.com
Address [1],[2] : Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India
http://dx.doi.org/10.22147/jusps-A/280506

Acceptance Date 28 September, 2016, Online Publication Date 2nd Oct. 2016

Abstract

A mixed quadrature rule, blending Clenshaw-Curtis five point rule in two dimensions and Gauss-Legendre three point rule in two dimensions, is formed. The mixed rule has been imposed with some test integrals and found to be more effective than that of its constituent rules.

Keywords: Clenshaw-Curtis quadrature rule $R^2_{CC_5}(f)$, Gauss-Legendre three point rule $R^2_{GL_3}(f)$, mixed quadrature rule $R^2_{CC_5GL_3}(f)$.

Subject classification: 65D30, 65D32

1 Introduction

Real definite integrals of the type

$$I(f)=\int_c^d\int_a^b f(x,y)\,dxdy=\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy \tag{1.1}$$

in two dimensions, where $f(x,y)$ is defined over the domain $[-1,1]\times[-1,1]$, have been successfully approximated by several authors[6,7,8,9]. Some of the authors[14, 15] also used mixed quadrature rule to evaluate integrals of the type (1.1). The mixed quadrature rule involves construction of symmetric quadrature rule of higher precision as a linear/ convex combination of two other rules of equal lower precision. Several authors [1, 2, 3, 10, 11, 12, 13] also successfully applied mixed quadrature rule to evaluate real definite integrals in one dimension.

If we consider a Gauss-Legendre rule and a Clenshaw-Curtis rule in two dimensions having same precision, Clenshaw-Curtis rule is better than Gauss-Legendre rule. An n-point Gaussian rule is of precision $2n-1$, while the precision of an n-point Clenshaw-Curtis rule is n. In general, Gauss type rule is of higher precision than that of Clenshaw-Curtis type when same abscissae are used.

In this paper, taking the advantage of the fact that Gauss-Legendre 3-point rule and Clenshaw-Curtis 5-point rule in two dimensions are of same precision (*i.e.*, precision 5) we formed a mixed quadrature rule of higher precision (*i.e.*, precision 7) taking linear combination of these rules. The mixed rule so formed has been tested on different integrals giving better results than Clenshaw-Curtis quadrature rule.

2 The Clenshaw-Curtis Quadrature Rule :

The Clenshaw-Curtis method[4] essentially approximates a function $f(t)$ over any interval $[\alpha-h,\alpha+h]$ using the Chebyshev polynomials $T_r(x)$ of degree n, *i.e.*,

$$f(t)=F(x)=\sum_{r=0}^{n}{}' a_r T_r(x) \qquad (-1\le x\le 1) \tag{2.1}$$

where a_r are the expansion coefficients and $\sum'$ denotes a finite sum whose first term is to be halved before beginning to sum. That is,

$$F(x)=\frac{1}{2}a_0T_0(x)+a_1T_1(x)+\cdots+a_nT_n(x) \tag{2.2}$$

Collocating with $f(\alpha+hx)$ at the $n+1$ points

$$x_i=\cos\left(\frac{i\pi}{n}\right), \qquad (i=0,1,\cdots,n) \tag{2.3}$$

One can evaluate the expansion coefficients a_r.

The Chebyshev polynomials $T_r(x_i)$ can be expressed as

$$T_r(x_i)=\cos(r\cos^{-1}(x_i)), \qquad r\ge 0 \tag{2.4}$$

$$=\cos\left(\frac{ri\pi}{n}\right)$$

Then

$$\sum_{i=0}^{n}{}'' f(\alpha+hx_i)T_r(x_i)=\sum_{i=0}^{n}{}''\sum_{k=0}^{n}{}' a_kT_k(x_i)T_r(x_i)$$

$$=\sum_{k=0}^{n}{}' a_k\sum_{i=0}^{n}{}''\cos\left(\frac{ki\pi}{n}\right)\cos\left(\frac{ri\pi}{n}\right) \tag{2.5}$$

The notation $\sum''$ means that the first and last terms are to be halved before summation begins. The orthogonality of the cosine function [5] with respect to the points $x_i = \cos\left(\frac{i\pi}{n}\right)$ is expressed by

$$\sum_{i=0}^{n}{}'' \cos\left(\frac{ki\pi}{n}\right)\cos\left(\frac{ri\pi}{n}\right) = \begin{cases} n, & r = k = 0 \ \ or \ \ n \\ \frac{n}{2}, & 0 < r = k < n \\ 0, & r \neq k \end{cases} \tag{2.6}$$

Substituting equation (2.6) into equation (2.5), we obtain

$$\sum_{i=0}^{n}{}'' f(\alpha + hx_i)T_r(x_i) = \begin{cases} \frac{n}{2}a_r, & 0 \le r = k \le n \\ na_r, & r = k = n \\ 0, & r \neq k \end{cases}$$

Hence

$$a_r = \begin{cases} \frac{2}{n}\sum_{i=0}^{n}{}'' f(\alpha + hx_i)T_r(x_i), & (r = 0,1,\cdots,n-1) \\ \frac{1}{n}\sum_{i=0}^{n}{}'' f(\alpha + hx_i)T_r(x_i), & (r = n) \end{cases} \tag{2.7}$$

Denoting the integral of $f(t)$ over the interval $[\alpha - h, \alpha + h]$ by I and replacing t by $\alpha + hx$ in equation, we get

$$I(f) = h\int_{-1}^{1} f(\alpha + h)dx$$

Assuming

$$I \approx I_n$$

we write

$$I_n = h\int_{-1}^{1}\sum_{i=0}^{n}{}' a_r T_r(x)dx$$

$$= h\sum_{i=0}^{n}{}' a_r \int_{-1}^{1} T_r(x)dx$$

Substituting the values of a_r (as given in equation (2.7)), we obtain

$$I_n(f) = {\sum_{r=0}^{n}}'' \frac{2}{n} {\sum_{i=0}^{n}}'' f(\alpha + hx_i) T_r(x_i) \int_{-1}^{1} T_r(x)dx$$

Since

$$\int_{-1}^{1} T_r(x)dx = \begin{cases} \dfrac{-2}{r^2-1}, & r = even \\ 0, & r = odd \end{cases} \tag{2.8}$$

we get

$$I_n = h {\sum_{r=0}^{n}}'' \frac{2}{n} w_i f(\alpha + hx_i) \tag{2.9a}$$

where

$$w_i = h \sum_{\substack{r=0 \\ r=even}}^{n}{}'' \frac{1}{r^2-1} T_r(x_i) \qquad (i = 0,1,\cdots,n) \tag{2.9b}$$

with $n = 4$

$$I_4(f) = \frac{h}{15}\left[f(\alpha - h) + 8f\left(\alpha - \frac{h}{\sqrt{2}}\right) + 12f(\alpha) + 8f\left(\alpha + \frac{h}{\sqrt{2}}\right) + f(\alpha + h)\right] \tag{2.10}$$

3. Construction of The Mixed Quadrature Rule of Precision Seven in Two Dimensions

The Clenshaw-Curtis five point rule in one dimension is

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{CC_5}(f) = \frac{1}{15}\left[f(-1) + 8f\left(-\frac{1}{\sqrt{2}}\right) + 12f(0) + 8f\left(\frac{1}{\sqrt{2}}\right) + f(1)\right] \tag{3.1}$$

So the Clenshaw-Curtis five point rule in two dimensions is

$$I(f) = \int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \approx R^2_{CC_5}(f)$$

$$= \frac{1}{15^2}\left[f(-1,-1) + 8f\left(-1,-\frac{1}{\sqrt{2}}\right) + 12f(-1,0) + 8f\left(-1,\frac{1}{\sqrt{2}}\right) + f(-1,1)\right]$$

$$+ \frac{8}{15^2}\left[f\left(-\frac{1}{\sqrt{2}},-1\right) + 8f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) + 12f\left(-\frac{1}{\sqrt{2}},0\right) + 8f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) + f\left(-\frac{1}{\sqrt{2}},1\right)\right]$$

$$+ \frac{12}{15^2}\left[f(0,-1) + 8f\left(0,-\frac{1}{\sqrt{2}}\right) + 12f(0,0) + 8f\left(0,\frac{1}{\sqrt{2}}\right) + f(0,1)\right]$$

$$+ \frac{8}{15^2}\left[f\left(\frac{1}{\sqrt{2}},-1\right) + 8f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) + 12f\left(\frac{1}{\sqrt{2}},0\right) + 8f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}},1\right)\right]$$

$$+\frac{1}{15^2}\left[f(1,-1)+8f\left(1,-\frac{1}{\sqrt{2}}\right)+12f(1,0)+8f\left(1,\frac{1}{\sqrt{2}}\right)+f(1,1)\right] \tag{3.2}$$

and the Gauss-Legendre three point rule in one dimension is

$$I(f)=\int_{-1}^{1}f(x)dx\approx R_{GL_3}(f)=\frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{3.3}$$

So the Gauss-Legendre three point rule in two dimension is

$$I(f)=\int_{-1}^{1}\int_{-1}^{1}f(x,y)dxdy\approx R_{GL_3}^2(f)$$

$$=\frac{5}{9^2}\left[5f\left(-\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(-\sqrt{\frac{3}{5}},0\right)+5f\left(-\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right]$$

$$+\frac{8}{9^2}\left[5f\left(0,-\sqrt{\frac{3}{5}}\right)+8f(0,0)+5f\left(0,\sqrt{\frac{3}{5}}\right)\right]$$

$$+\frac{5}{9^2}\left[5f\left(\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(\sqrt{\frac{3}{5}},0\right)+5f\left(\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right] \tag{3.4}$$

Let $E_{CC_5}^2(f)$ and $E_{GL_3}^2(f)$ denote the error terms in approximating the integral $I(f)$ by the rules (3.2) and (3.4) respectively. Let

$$I(f)=R_{CC_5}^2(f)+E_{CC_5}^2(f) \tag{3.5}$$

$$I(f)=R_{GL_3}^2(f)+E_{GL_3}^2(f) \tag{3.6}$$

Using Maclaurin's expansion of functions in two variables in equations (3.2) and (3.4) we get

$$E_{CC_5}^2(f)=\frac{1}{18900}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{1}{907200}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]$$

$$+\frac{1}{113400}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\cdots \tag{3.7}$$

$$E_{GL_3}^2(f)=\frac{1}{7875}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{11}{2835000}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]$$

$$+\frac{1}{47250}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\cdots \tag{3.8}$$

This shows that the rules (3.2) and (3.4) are of precision 5.

Now multiplying the equations (3.5) and (3.6) by $\frac{1}{5}$ and $-\frac{1}{12}$ respectively, and then adding the resulting equations we obtain

$$I(f)=\frac{1}{7}\left[12R^2_{CC_5}(f)-5R^2_{GL_3}(f)\right]+\frac{1}{7}\left[12E^2_{CC_5}(f)-5E^2_{GL_3}(f)\right]$$

or $$I(f)=R^2_{CC_5GL_3}(f)+E^2_{CC_5GL_3}(f) \tag{3.9}$$

where $$R^2_{CC_5GL_3}(f)=\frac{1}{7}\left[12R^2_{CC_5}(f)-5R^2_{GL_3}(f)\right] \tag{3.10}$$

This is the desired mixed quadrature rule of precision seven for approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$\begin{aligned} E^2_{CC_5GL_3}(f) &= \frac{1}{7}\left[12E^2_{CC_5}(f)-5E^2_{GL_3}(f)\right] \\ &= -\frac{1}{1134000}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\cdots \end{aligned} \tag{3.11}$$

The rule (3.10) may be called as a mixed type rule as it is constructed from two different types of rules of the same precision (*i.e.,* precision 5).

4 Error Analysis :

An asymptotic error estimate and error bound of the rule (3.10) are given in theorems (4.1) and (4.2) respectively.

Theorem 4.1: Let $f(x,y)$ be a sufficiently differentiable function in the closed interval $[-1,1]\times[-1,1]$. Then the error $E^2_{CC_5GL_3}(f)$ associated with the rule $R^2_{CC_5GL_3}(f)$ is given by

$$\left|E^2_{CC_5GL_3}(f)\right|\cong\frac{1}{1134000}\left|\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]\right|$$

Proof. Follows immediately from equation (3.11). □

Theorem 4.2: The bound for the truncation error $E^2_{CC_5GL_3}(f)=I(f)-R^2_{CC_5GL_3}(f)$

$$\left|E^2_{CC_5GL_3}(f)\right|\le\frac{M}{11025}\left|\xi_2-\xi_1\right|\times\left|\eta_2-\eta_1\right|$$

where $M=\max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|$

Proof. We have from (3.7) and (3.8)

$$E^2_{CC_5}(f)\cong\frac{1}{18900}\left[f_{6,0}(\xi_2,\eta_2)+f_{0,6}(\xi_2,\eta_2)\right],\quad (\xi_2,\eta_2)\in[-1,1]\times[-1,1]$$

$$E^2_{GL_3}(f)\cong\frac{1}{7875}\left[f_{6,0}(\xi_1,\eta_1)+f_{0,6}(\xi_1,\eta_1)\right],\quad (\xi_1,\eta_1)\in[-1,1]\times[-1,1]$$

We know

$$\begin{aligned} E^2_{CC_5GL_3}(f) &= \frac{1}{7}\left[12E^2_{CC_5}(f)-5E^2_{GL_3}(f)\right] \\ &\cong\frac{1}{7}\left[\frac{1}{1575}\left\{f_{6,0}(\xi_2,0)+f_{0,6}(0,\eta_2)\right\}-\frac{1}{1575}\left\{f_{6,0}(\xi_1,0)+f_{0,6}(0,\eta_1)\right\}\right] \end{aligned}$$

$$= \frac{1}{11025}\left[\left\{f_{6,0}(\xi_2,0)+f_{0,6}(0,\eta_2)\right\}-\left\{f_{6,0}(\xi_1,0)+f_{0,6}(0,\eta_1)\right\}\right]$$

$$= \frac{1}{11025}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]dxdy \quad (\text{assuming } \xi_1<\xi_2 \text{ and } \eta_1<\eta_2)$$

Hence $\left|E^2_{CC_5GL_3}(f)\right| \cong \left|\frac{1}{11025}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]dxdy\right|$

$$\le \frac{1}{11025}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|dxdy$$

$\because$ f(x, y) is defined on a closed and bounded rectangle $[-1,1]\times[-1,1]$, hence compact and so f(x, y) attains its maximum over the domain $[-1,1]\times[-1,1]$.

so $\left|E^2_{CC_5GL_3}(f)\right| \le \frac{1}{11025}M\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}dxdy,$ where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|$

$$= \frac{M}{11025}\left|(\xi_2-\xi_1)\times(\eta_2-\eta_1)\right|$$

which gives only a theoretical error bound as (ξ_1,η_1) and (ξ_2,η_2) are unknown points in $[-1,1]\times[-1,1]$. It shows that the error in the approximation will be less if the points (ξ_1,η_1), (ξ_2,η_2) get closed to each other.

Corollary 4.1 : The error bound for the truncation error $E^2_{CC_5GL_3}(f)$ is given by

$$\left|E^2_{CC_5GL_3}(f)\right| \le \frac{4M}{11025}$$

Proof. we know from theorem (4.2) that

$$\left|E^2_{CC_5GL_3}(f)\right| \le \frac{M}{11025}\left|(\xi_2-\xi_1)\times(\eta_2-\eta_1)\right|, \quad (\xi_1,\eta_1),(\xi_2,\eta_2),\in[-1,1]\times[-1,1]$$

where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|$

choosing $|\xi_2-\xi_1|\le 2$ and $|\eta_2-\eta_1|\le 2$

weget $\left|E^2_{CC_5GL_3}(f)\right| \le \frac{4M}{11025}$

5 Numerical Verification :

Table 1. Comparison of the Mixed Quadrature Rule with Clenshaw-Curtis 5-point Rule in Approximation of some Real Definite Integrals in two dimensions

Integrals	Exact Value	Approximate Value			Error Approximated		
		$R^2_{CC_5}(f)$	$R^2_{GL_3}(f)$	$R^2_{CC_5GL_3}(f)$	$E^2_{CC_5}(f)$	$E^2_{GL_3}(f)$	$E^2_{CC_5GL_3}(f)$
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}\,dxdy$	5.524391382167262	5.524264412485792	5.5240367316988	5.524393508320651	0.000126969681470	0.000307703850274	$-2.126153388\times10^{-6}$
$\int_{0}^{1}\int_{0}^{x^2} \frac{y}{e^x}\,dydx$	0.04391861928124	0.043907646528054	0.043892518086722	0.043918452557576	0.0000105154000704	0.000025643841401	-2.90629452×10^{-7}
$\int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)}\,dydx$	2.2309851414041345	2.238065053066174	2.24604	2.23268662399156	-0.007079911662039	-0.015054858595865	-0.001383520995021
$\int_{1}^{1}\int_{\frac{y^2}{2}}^{\sqrt{y}}\,dxdy$	$\frac{1}{2}$	0.491259503108319	0.502512967232805	0.498221314447972	0.00874096891680	-0.002512967232805	0.001778685552027

6 Conclusion

Above examples give a clear picture about the effectiveness of imposing mixed quadrature rule rather than its constituent rules. The mixed quadrature rule gives better result in comparison its constituent rules with the precision enhancement and also it has been tested this mixed rule we have applied here also very much effective than that of some previous paper[15] which is the basic intuition behind this paper.

References

1. Das, R. N., and Pradhan, G., *"A Mixed Quadrature Rule for Approximate Evaluation of Real Definite Integrals"*, Int. J. Math. Educ. Sci. Technol., 1996, Vol.-27, No.-2, 279-283 (1996).
2. Das, R. N., and Pradhan, G., *"A Mixed Quadrature Rule for Numerical Integration of Analytic Functions"*, Bulletin of Cal. Math. Soc., *89,* 37-42 (1997).
3. Dash, R. B., and Mohanty, S. K., *"A Mixed Quadrature Rule for Numerical Integration of Analytic Functions"*, Bulletin, Pure and Applied Sciences, Vol.-27E (No.2), 2008, 369-372 (2008).
4. Oliver, J., *"A-Doubly-Adaptive Clenshaw-Curtis Quadrature Method"*, Computing centre, University of Essex, Wivenhoe park, Colchester, Essex (1971).
5. Atkinson, K. E., *"An Introduction to Numerical Analysis"*, 2nd ed., John Wiley, (2001).
6. Jain, M. K., Iyengar, S. R. K. and Jain, R. K., *"Numerical Methods for Scientific and Engineering Computation"*, Fourth Edition, New Age International Publishers.
7. Acharya, B. P. and Mohapatra, T., *"Numerical Evaluation of Complex Multiple Integrals of Analytic Functions"*, Facta Univ. (NIS), Ser. Math. Inform., *4,* 83-88 (1989).
8. Acharya, B. P. and Das, R. N., *"Approximate Evaluation of Multiple Complex Integrals of Analytic Functions"*, Computing, *30,* 279-283. http://dx.doi.org/10.1007/bf02253899 (1983).
9. Acharya, M., Nayak, M. M. and Acharya, B. P., *"Numerical Evaluation of Complex Double Integrals of Analytic Functions"*, Proc. Int. Conf. on CAMIST, 213-218 (2010).
10. Dash, Rajani B. and Das, D., *"A Mixed Quadrature Rule by Blending Clenshaw-Curtis and Gauss-Legendre Quadrature Rules for Approximation of Real Definite Integrals in Adaptive Environment"*, proceedings of the international multi-conference of Engineer and computer scientists, Vol-I, 202-205, Hong-Kong (2011).
11. Dash, Rajani B. and Das, D., *"On the Use of Mixed Quadrature in Adaptive Quadrature Routines"*, Global Journal of Mathematics and Mathematical Sciences, Vol. *2,* No.-1, 45-56, RIP, New Delhi, (2012).
12. Pati, A., Dash, R. B. and Patra, P., *"A Mixed Quadrature Rule by Blending Clenshaw-Curtis and Gauss-Legendre Quadrature Rules For Approximate Evaluation of Real Definite Integrals"*, The Bulletin of Society for Mathematical Services and Standards, Sci. Press, Switzerland, Vol-*2,* 10-15, (2012).
13. Patra, P., Das. D, and Dash, R. B., *"An Adaptive Integration Scheme Using a Mixed Quadrature of Three Different Quadrature Rules"*, Malaya Journal of Mathematik *3(3),* 224-232 (2015).
14. Patra, P., Behera, D. and Dash, R. B., *"Evaluation of Real Definite Integrals by Using Mixed Quadrature Rules over a Triangular Domain"*, Journal of Advances in Mathematics, Vol-12, No.-2, 5945-5948, (2016).
15. Jena, S. R. and Mishra, S. C., *"Mixed Quadrature Rule for Double Integrals of Simpson's $\frac{1}{3}^{rd}$ and Gauss Legendre Two Point Rule in Two Variables"*, Global Journal of Science Frontier Research: F Mathematics and Decision Sciences, Vol-16, Issue 1, Global Journals Inc. (USA) (2016).