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k-Symmetric Circulant, s-Symmetric Circulant and s-k Symmetric Circulant Matrices

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Abstract

The basic concepts and theorems of k-symmetric Circulant, s-symmetric Circulant and s-k-symmetric Circulant matrices are introduced with examples.

Key words : k-symmetric Circulant matrix, s-symmetric Circulant matrix and s-k-symmetric Circulant matrix.

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Introduction

The concept of s-symmetric matrices, k-symmetric matrices and of s-k symmetric matrices was introduced in ¹⁻⁴. Some properties of symmetric matrices given in ^{5,6}. In this paper, our intention is to define s-symmetric circulant matrices, k-symmetric circulant matrices and s-k symmetric circulant matrices also we discussed some results on symmetric circulant matrices.

Preliminaries and Notations :

C is circulant matrix, C^T is called Transpose of C, C^S is called secondary transpose of C. Let k be a fixed product of disjoint transposition in S_n and 'K' be the permutation matrix associated with K, V is a permutation matrix with units in the secondary diagonal. Clearly K and V are satisfies the following properties. $K^2 = I$, $K^T = K$, $V^2 = I$, $V^T = V$

*Definitions and Theorems :**Definition : 1*

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For any given $c_0, c_1, c_2, \dots, c_{n-1} \in \mathbb{R}^{n \times n}$ the circulant matrix $C = (c_{i,j})_{n \times n}$ is defined by $(c_{i,j}) = c_{j-1 \pmod{n}}$

$$\begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{bmatrix}$$

Definition : 2 A matrix $C \in \mathbb{R}^{n \times n}$ is said to be symmetric circulant matrix if $C = C^T$

$$C_{ij} = C_{ji} \text{ for all } i, j = 1, 2, 3, \dots, n$$

Theorem : 1 Let $C \in \mathbb{R}^{n \times n}$ is k-symmetric circulant matrix then $C = K C^T K$.

Proof: $K C^T K = K C K$ where $C^T = C$
 $= C K K$ where $K C = C K$
 $= C K^2 = C$

Theorem : 2 Let $C \in \mathbb{R}^{n \times n}$ is k-symmetric circulant matrix then $C^T = K C K$.

Proof: $K C K = K C^T K$ where $C = C^T$
 $= C^T K K$ where $K C^T = C^T K$
 $= C^T K^2 = C^T$

Theorem : 3 If C_1 and C_2 are k-symmetric Circulant matrices then $C_1 C_2$ is also k-symmetric Circulant matrix

Proof: Let C_1 and C_2 are k-symmetric circulant matrices if $C_1 = K C_1^T K$ and $C_2 = K C_2^T K$.

Since C_1^T and C_2^T are also k-symmetric Circulant matrices then $C_1^T = K C_1 K$ and $C_2^T = K C_2 K$.

To prove $C_1 C_2$ is k-symmetric Circulant matrix

We will show that $C_1 C_2 = K (C_1 C_2)^T K$

$$\begin{aligned} \text{Now } K (C_1 C_2)^T K &= K C_2^T C_1^T K \\ &= K [(K C_2 K)(K C_1 K)] K \text{ where } C_1^T = K C_1 K \text{ and } C_2^T = K C_2 K. \\ &= K^2 C_2 K^2 C_1 K^2 \\ &= C_2 C_1 \quad \text{where } K^2 = I \\ &= C_1 C_2 \quad \text{where } C_2 C_1 = C_1 C_2 \end{aligned}$$

Theorem : 4 If C is k-symmetric Circulant matrices and K is the permutation matrix, $k = (12)$ then $K C$ is also k-symmetric Circulant matrix.

Proof: A matrix $C \in \mathbb{R}^{n \times n}$ is said to be k-symmetric Circulant matrix if $C = K C^T K$

Since C^T is also k-symmetric Circulant matrices then $C^T = K C K$

To prove $K C$ is K-symmetric Circulant matrix

We will show that $K C = K (K C)^T K$

$$\begin{aligned} \text{Now } K (K C)^T K &= K (C^T K^T) K \quad \text{where } (K C)^T = C^T K^T \\ &= K C^T \quad \text{where } K^T K = I \\ &= K C \quad \text{where } K C^T = K C \end{aligned}$$

Theorem : 5 If $C \in \mathbb{R}^{n \times n}$ is k-symmetric circulant matrix then $C C^T$ is also k-symmetric Circulant matrix

Proof : A matrix $C \in \mathbb{R}^{n \times n}$ is said to be k-symmetric Circulant matrix if $C = K C^T K$

Since C^T is also k-symmetric Circulant matrices then $C^T = K C K$

We will show that $C C^T = K (C C^T)^T K$

$$\begin{aligned} \text{For that, } K (C C^T)^T K &= K [(C^T)^T C^T] K \quad \text{where } (K C)^T = C^T K^T \\ &= K (C C^T) K \quad \text{where } (C^T)^T = C \\ &= (C C^T) K K \quad \text{where } K C = C K \end{aligned}$$

$$= (C C^T) K^2 \quad \text{where } KK=K^2$$

$$= (C C^T)$$

Theorem : 6 If $C \in R^{n \times n}$ is k-symmetric circulant matrix then $C \pm C^T$ is also k-symmetric Circulant matrix

Proof: A matrix $C \in R^{n \times n}$ is said to be k-symmetric Circulant matrix if $C = KC^T K$

Since C^T is also k- symmetric Circulant matrices then $C^T = KCK$

We will show that , $C + C^T = K (C + C^T)^T K$

$$\text{For that, } K (C + C^T)^T K = K [(C^T)^T + C^T] K \quad \text{where } (C_1 + C_2)^T = (C_1^T + C_2^T)$$

$$= K (C + C^T) K \quad \text{where } (C^T)^T = C$$

$$= (C + C^T) KK \quad \text{where } KC = CK$$

$$= (C + C^T) K^2$$

$$= (C + C^T)$$

Result

Let C_1 and C_2 are k-symmetric Circulant matrices for the following conditions are holds

- [i] $C_1 \pm C_2$ is also k- symmetric Circulant matrix
- [ii] $C_1 C_2 = C_2 C_1$
- [iii] $(C_1^T C_2 C_1)$ and $(C_2^T C_1 C_2)$ are also k- symmetric Circulant matrices
- [iv] $\text{Adj } C_1$ also k- symmetric Circulant matrix
- [v] $C_1(\text{Adj } C_1)$ is also k- symmetric Circulant matrix

Example: 1

$$\text{Let } C_1 = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}; K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(i) K (C_1 C_2)^T K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 26 & 23 \\ 23 & 26 \end{pmatrix} = C_1 C_2$$

$$(ii) K (C_1^T C_2 C_1)^T K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 170 & 173 \\ 173 & 170 \end{pmatrix} = C_1^T C_2 C_1$$

Definition: 3 A matrix $C \in R^{n \times n}$ is said to be s-symmetric circulant matrix if $C^S = VC^T V$ where V is a permutation matrix with units in the secondary diagonal

Theorem: 7 Let $C \in R^{n \times n}$ is s-symmetric circulant matrix then $C^S = VC^T V$

Proof: If C is s-symmetric circulant matrix then $C^S = VC^T V$ and C^T is s-symmetric circulant matrix then $C^T = VC^S V$

We will show that $C^S = VC^T V$

$$\text{For that } V C^T V = V (V C^S V) V = V^2 C^S V^2 \quad \text{where } C^T = VC^S V$$

$$= C^S \quad \text{where } V^2 = I$$

Theorem: 8 Let $C \in R^{n \times n}$ is s-symmetric circulant matrix then $C^T = VC^S V$

Proof: If C is s-symmetric circulant matrix then $C^S = VC^T V$

and C^T is s-symmetric circulant matrix then $C^T = VC^S V$

We will show that $C^T = VC^S V$

$$\text{For that } V C^S V = V (V C^T V) V$$

$$= V^2 C^T V^2 \quad \text{where } C^S = VC^T V$$

$$= C^T \quad \text{where } V^2 = I$$

Theorem : 9 If $C_1, C_2 \in R^{n \times n}$ are s-symmetric circulant matrices then $(C_1 \pm C_2)$ is s-symmetric circulant matrix

Proof: Let C_1 , and C_2 are s-symmetric circulant matrices if $C_1^S = VC_1^T V$ and $C_2^S = VC_2^T V$.

To prove $(C_1 \pm C_2)$ is s- symmetric circulant matrix

We will show that $(C_1 \pm C_2)^S = V (C_1 \pm C_2)^T V$

$$\text{Now } V (C_1 \pm C_2)^T V = V (C_1^T \pm C_2^T) V \quad \text{where } (C_1 \pm C_2)^T = (C_1^T \pm C_2^T)$$

$$= (V C_1^T \pm V C_2^T) V$$

$$= (V C_1^T V \pm V C_2^T V)$$

$$= (C_1^S \pm C_2^S) \quad \text{where } C_1^S = VC_1^T V \text{ and } C_2^S = VC_2^T V$$

Theorem:10 If C_1 and C_2 are s-symmetric Circulant matrices then $C_1 C_2$ is also s-symmetric Circulant matrix

Proof: Let C_1 , and C_2 are s-symmetric circulant matrices if $C_1^s = VC_1^T V$ and $C_2^s = VC_2^T V$.

Since C_1^T and C_2^T are also s-symmetric Circulant matrices then $C_1^T = VC_1^s V$ and $C_2^T = VC_2^s V$.

To prove $C_1 C_2$ is s-symmetric Circulant matrix

We will show that $(C_1 C_2)^s = V(C_1 C_2)^T V$

$$\begin{aligned} \text{Now } V(C_1 C_2)^T V &= V C_2^T C_1^T V \\ &= V [(VC_2^s V)(V C_1^s V)] V \text{ where } C_1^T = V C_1^s V \text{ and } C_2^T = V C_2^s V. \\ &= V^2 C_2^s V^2 C_1^s V^2 \\ &= C_2^s C_1^s \quad \text{where } V^2 = I \\ &= (C_1 C_2)^s \quad \text{where } C_2^s C_1^s = (C_1 C_2)^s \end{aligned}$$

Theorem:11 A matrix $C \in R^{n \times n}$ is said to be s-symmetric circulant matrix and V is a permutation matrix with units in the secondary diagonal then VC is also s-symmetric circulant matrix

Proof: Let C is s-symmetric circulant matrix then $C^s = VC^T V$ and $C^T = VC^s V$

To prove VC is s-symmetric circulant matrix

We will show that $(VC)^s = V(VC)^T V$

$$\begin{aligned} \text{For that, } V(VC)^T V &= V(C^T V^T) V \quad \text{where } (VC)^T = C^T V^T \\ &= V(VC^s V) V^T V \quad \text{where } C^T = VC^s V \\ &= V^2 C^s (VV^T V) \\ &= V^2 C^s V^s \quad \text{where } VV^T V = V^s \\ &= C^s V^s \quad \text{where } V^2 = I \\ &= (VC)^s \quad \text{where } C^s V^s = (VC)^s \end{aligned}$$

Theorem :12 If $C \in R^{n \times n}$ is s-symmetric circulant matrix then $C C^T$ is also s-symmetric Circulant matrix

Proof : A matrix C is said to be s-symmetric Circulant matrix if $C^s = VC^T V$

Since C^T is also k- symmetric Circulant matrices then $C^T = VC^s V$

We will show that, $(C C^T)^s = V(C C^T)^T V$

$$\begin{aligned} \text{For that, } V(C C^T)^T V &= V[V(C C^T)^s V] V \text{ where } (VC)^s = C^s V^s \\ &= V^2 (C C^T)^s V^2 \quad \text{where } VV = V^2 \\ &= (C C^T)^s \quad \text{where } V^2 = I \end{aligned}$$

Theorem : 13 If $C \in R^{n \times n}$ is s-symmetric circulant matrix then $C \pm C^T$ is also s-symmetric Circulant matrix

Proof: A matrix is said to be s-symmetric Circulant matrix if $C^s = VC^T V$

Since C^T is also k- symmetric Circulant matrices then $C^T = VC^s V$

We will show that, $(C \pm C^T)^s = V(C \pm C^T)^T V$

$$\begin{aligned} \text{For that, } V(C \pm C^T)^T V &= V[V[(C \pm C^T)^s V] V] \text{ where } C^s = VC^T V \\ &= V^2 (C \pm C^T)^s V^2 \quad \text{where } VV = V^2 \\ &= (C \pm C^T)^s \quad \text{where } V^2 = I \end{aligned}$$

Result

Let C_1 and C_2 are s-symmetric Circulant matrices for the following conditions are holds

- [i] $C_1 \pm C_2$ is also s- symmetric Circulant matrix
- [ii] $C_1 C_2 = C_2 C_1$
- [iii] $(C_1^T C_2 C_1)$ and $(C_2^T C_1 C_2)$ are also s- symmetric Circulant matrices
- [iv] $\text{Adj } C_1$ also s- symmetric Circulant matrix
- [v] $C_1(\text{Adj } C_1)$ is also s- symmetric Circulant matrix

Example : 2

$$\text{Let } C_1 = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} ; V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(i) \quad V(C_1 C_2)^T V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 26 & 23 \\ 23 & 26 \end{pmatrix} = (C_1 C_2)^s$$

$$(ii) \quad V(C_1^T C_2 C_1)^T V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 170 & 173 \\ 173 & 170 \end{pmatrix} = (C_1^T C_2 C_1)^s$$

Definition:4 A matrix $C \in R^{n \times n}$ is said to be s-k symmetric circulant matrix if

- (i) $C = KVC^T VK$
- (ii) $C^T = KVCVK$
- (iii) $C^S = VKC^T KV$
- (iv) $C^T = VKC^S KV$

where V is a permutation matrix with units in the secondary diagonal and K is the permutation matrix, $k = (12)$

Theorem :14 A matrix $C \in R^{n \times n}$ is said to be s-k symmetric circulant matrix then

- (i) $C^S = KVC^T VK$
- (ii) $C^T = KVC^S VK$
- (iii) $C^S = VKC^T KV$
- (iv) $C^T = VKC^S KV$

Proof:

- (i) $KVC^T VK = K(VC^T V)K$
 $= K(C^S)K$ where $C^S = VC^T V$
 $= K(C^T)^T K$ where $C^S = (C^T)^T$
 $= K(C)^T K$ where $C = C^T$
 $= C$ where $K(C)^T K = C$
 $= C^S$ where $C = C^S$
- (ii) $KVC^S VK = K(VC^S V)K$
 $= K(C^T)K$ where $C^T = VC^S V$
 $= C$ where $K(C)^T K = C$
 $= C^T$ where $C^T = C$
- (iii) $VKC^T KV = V(KC^T K)V$
 $= VC^T V$ where $C = KC^T K$
 $= VC^T V$ where $C = C^T$
 $= C^S$ where $V C^T V = C^S$
- (iv) $VKC^S KV = V(KC^S K)V$
 $= V[K(C^T)^T K]V$ where $(C^T)^T = C^S$
 $= VC^T V$ where $K(C^T)^T K = C^T$
 $= C^S$ where $V C^T V = C^S$
 $= (C^T)^T = C^T$ where $C^T = C$

Theorem :15 If $C_1, C_2 \in R^{n \times n}$ are s-k symmetric circulant matrices then $(C_1 \pm C_2)$ is s-k symmetric circulant matrix

Proof: Let C_1 , and C_2 are s-k symmetric circulant matrices if $C_1 = KV C_1^T VK$ and $C_2 = KVC_2^T VK$.

To prove $(C_1 \pm C_2)$ is s-k symmetric circulant matrix

We will show that $(C_1 \pm C_2) = KV(C_1 \pm C_2)^T VK$

$$\begin{aligned} \text{Now } KV(C_1 \pm C_2)^T VK &= K[V(C_1^T \pm C_2^T)V]K \quad \text{where } (C_1 \pm C_2)^T = (C_1^T \pm C_2^T) \\ &= K(C_1 \pm C_2)^S K \quad \text{where } V C^T V = C^S \\ &= (C_1 \pm C_2)^T \quad \text{where } KC^S K = C^T \\ &= C_1 \pm C_2 \quad \text{where } C^T = C \end{aligned}$$

Theorem : 16 If $C_1, C_2 \in R^{n \times n}$ are s-k symmetric circulant matrices then $(C_1 C_2)$ is s-k symmetric circulant matrix

Proof: Let C_1 , and C_2 are s-k symmetric circulant matrices if $C_1 = KV C_1^T VK$ and $C_2 = KVC_2^T VK$.

And C_1^T , and C_2^T are s-k symmetric circulant matrices if $C_1^T = KV C_1^T VK$ and $C_2^T = KVC_2^T VK$.

To prove $(C_1 C_2)$ is s-k symmetric circulant matrix

We will show that $(C_1 C_2) = KV(C_1 C_2)^T VK$

$$\begin{aligned} \text{Now } KV(C_1 C_2)^T VK &= K[V(C_1 C_2)^T V]K \\ &= K(C_1 C_2)^S K \quad \text{where } V C^T V = C^S \\ &= (C_1 C_2)^T \quad \text{where } KC^S K = C^T \end{aligned}$$

$$= C_1 C_2 \quad \text{where } C^T = C$$

Theorem :17 If $C \in R^{n \times n}$ is s-k symmetric circulant matrix then $C C^T$ is also s-k symmetric Circulant matrix

Proof : Let C is s-k symmetric circulant matrices if $C = KV C^T VK$

And C^T is s-k symmetric circulant matrices if $C^T = KV C VK$

We will show that, $(C C^T) = KV (C C^T)^T VK$

For that, $KV (C C^T)^T VK = K [V (C C^T)^T V] K$

$$= K (C C^T)^S K \quad \text{where } V C^T V = C^S$$

$$= (C C^T)^T \quad \text{where } K C^S K = C^T$$

$$= C C^T \quad \text{where } C^T = C$$

Theorem:18 If $C \in R^{n \times n}$ is s-k symmetric circulant matrix then $C \pm C^T$ is also s-k symmetric Circulant matrix

Proof : Let C is s-k symmetric circulant matrices if $C = KV C^T VK$

And C^T is s-k symmetric circulant matrices if $C^T = KV C VK$

To prove $(C \pm C^T)$ is s-k symmetric circulant matrix

We will show that $(C \pm C^T) = KV (C \pm C^T)^T VK$

Now $KV (C \pm C^T)^T VK = K [V (C \pm C^T)^T V] K$

$$= K (C \pm C^T)^S K \quad \text{where } V C^T V = C^S$$

$$= (C \pm C^T)^T \quad \text{where } K C^S K = C^T$$

$$= C \pm C^T \quad \text{where } C^T = C$$

Result

Let C_1 and C_2 are s-k-symmetric Circulant matrices for the following conditions are holds

[i] $C_1 \pm C_2$ is also s-k- symmetric Circulant matrix [ii] $C_1 C_2 = C_2 C_1$

[iii] $(C_1^T C_2 C_1)$ and $(C_2^T C_1 C_2)$ are also s-k- symmetric Circulant matrices

[iv] $\text{Adj } C_1$ also s-k- symmetric Circulant matrix [v] $C_1 (\text{Adj } C_1)$ is also s-k- symmetric Circulant matrix

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