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Dual Pell Quaternions

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Abstract

In this paper, we defined dual Pell quaternions. Also, we investigated the relations between dual Pell quaternions which connected with Pell and Pell-Lucas numbers. Furthermore, we gave the Binet formulas and Cassini-like identities for these quaternions.

Key words: Pell number, Pell-Lucas number, Pell quaternion, dual Pell quaternion.

1. **Introduction**

In 1843, Hamilton[1] introduced the set of quaternions which can be represented as

$$H = \{q = q_0 + i\, q_1 + j\, q_2 + k\, q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\} \tag{1.1}$$

where

$$i^2 = j^2 = k^2 = -1\,,\ i\,j = -j\,i = k\,,\ j\,k = -k\,j = i\,,\ k\,i = -i\,k = j\,.$$

Several authors worked on different quaternions and their generalizations[12-22]. Also, some authors worked on dual quaternions and their generalizations[2-6] as follows:

In 2006, Majernik[2] defined dual quaternions as follows:

$$H_{\mathbb{D}} = \left\{ \begin{array}{l} Q = a + b\,i + c\,j + d\,k \mid a, b, c, d \in \mathbb{R},\ i^2 = j^2 = k^2 = 0 \\ \quad i\,j\,k = 0,\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0 \end{array} \right\}. \tag{1.2}$$

In 2009, Ata and Yayl1[3] defined dual quaternions with dual numbers[1] coefficient as follows:

$$H(\mathbb{D}) = \{Q = A + Bi + Cj + Dk \mid A, B, C, D \in \mathbb{D},\ i^2 = j^2 = k^2 = -1 = ijk\} \tag{1.3}$$

In 2014, Nurkan and Güven[4] defined dual Fibonacci quaternions as follows:

$$HD = \{\tilde{O}_n = \tilde{F}_n + i\,\tilde{F}_{n+1} + j\,\tilde{F}_{n+2} + k\,\tilde{F}_{n+3} \mid \tilde{F}_n = F_n + \epsilon F n{+}1,\ \epsilon^2 = 0,\ \epsilon \neq 0\}, \quad (1.4)$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\ i\,j = -j\,i = k\,,\ j\,k = -k\,j = i\,,$$
$$k\,i = -i\,k = j\,,$$

$n \geq$ and $\tilde{O}_n = Q_n + \varepsilon\, Q_{n+1}$.

Essentially, these quaternions in equations (1.3) and (1.4) must be called dual numbers coefficient's quaternion and dual numbers coefficient's Fibonacci quaternions, respectively.

For more details on dual quaternions, see[5]. It is clear that $H(\mathbb{D})$ and $H_{\mathbb{D}}$ are different sets.

In 2+016, Yüce and Torunbalc1 Ayd1n[6] defined dual Fibonacci quaternions as follows[6]:

$$H_{\mathbb{D}} = \{Q_n = F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3} \mid F_n\ n\text{-th Fibonacci number}\}, \quad (1.5)$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

In 1971, Horadam studied on the Pell and Pell-Lucas sequences and he gave Cassini-like formula as follows[7]:

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n \quad (1.6)$$

and Pell identities

$$\begin{cases} P_rP_{n+1} + P_{r-1}P_n = P_{n+r}, \\ P_n(P_{n+1} + P_{n-1}) = P_{2n}, \\ P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n, \\ P_n^2 + P_{n+1}^2 = P_{2n+1}, \\ P_n^2 + P_{n+3}^2 = 5(P_{n+1}^2 + P_{n+2}^2), \\ P_{n+a}P_{n+b} - P_nP_{n+a+b} = (-1)^n P_nP_{n+a+b}, \\ P_{-n} = (-1)^{n+1}P_n. \end{cases} \quad (1.7)$$

and in 1985, Horadam and Mohan obtained Cassini-like formula as follows[8]

$$q_{n+1}q_{n-1} - q_n^2 = 8\,(-1)^{n+1}. \quad (1.8)$$

First the idea to consider Pell quaternions it was suggested by Horadam in paper[9].

In 2016, Çimen and İpek introduced the Pell quaternions and the Pell-Lucas quaternions and gived properties of them[10] as follows:

$$QP_n = \{QP_n = P_n\,e_0 + P_{n+1}\,e_1 + P_{n+2}\,e_2 + P_{n+3}\,e_3 \mid P_n\ n\text{-th Pell number}\} \quad (1.9)$$

where

$$e_1^2 = e_2^2 = e_3^2 = -1,\ e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2.$$

In 2016, Anetta and Iwona introduced the Pell quaternions and the Pell octanions[11] as follows:

$$R_n = P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3} \quad (1.10)$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\ i\,j = -j\,i = k,\ j\,k = -k\,j = i,\ k\,i = -i\,k = j\,.$$

Furthermore, Anetta and Iwona introduced the matrix generator for Pell and Pell-Lucas quaternions as follows

$$R(n) = \begin{bmatrix} R_n & R_{n-1} \\ R_{n-1} & R_{n-2} \end{bmatrix} \quad (1.11)$$

In this paper, we will define dual Pell quaternions as follows

$$P_D = \{D_n^P = P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}.\mid P_n\ n\text{-th Pell number}\} \quad (1.12)$$

---

[1]Dual number: $A = a + \varepsilon b,\ a, b \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0.$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\ ,\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\ . \tag{1.13}$$

Also, we will give Binet Formula and Cassini identities for dual Pell quaternions.

2. Dual Pell Quaternions :

We can define dual Pell quaternions by using the Pell numbers. The nth Pell numbers is defined by

$$P_n = P_{n-1} + P_{n-2}\ ,\ P_0 = 0,\ \ P_1 = 1. \tag{2.1}$$

With the same analogy we can define the dual Pell quaternions as follows

$$P_D = \{D_n^P = P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}\ .|\ P_n\ n\text{-}th \text{ Pell number}\}, \tag{2.2}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\ ,\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\ . \tag{2.3}$$

Also, we can define the dual Pell-Lucas quaternion as follows:

$$p_D = \{D_n^P = q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}\ .|\ q_n\ n\text{-}th \quad \text{Pell-Lucas number}\}, \tag{2.4}$$

$$i^2 = j^2 = k^2 = i\,j\,k = 0\ ,\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\ .$$

Let $D_n^{P_1}$ and $D_n^{P_2}$ be n-th terms of the dual Pell quaternion sequences $(D_n^{P_1}\ )$and $(D_n^{P_2})$ such that

$$D_n^{P_1} = P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3} \tag{2.5}$$

and

$$D_n^{P_2} = K_n + iK_{n+1} + j\,K_{n+2} + k\,K_{n+3} \tag{2.6}$$

Then, the addition and subtraction of the dual Pell quaternions is defined by

$$\begin{aligned} D_n^{P_1} \pm D_n^{P_2} = &\ (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &\pm (K_n + i\,K_{n+1} + j\,K_{n+2} + k\,K_{n+3}) \\ &= (P_n \pm K_n) + i\,(P_{n+1} \pm K_{n+1}) + j\,(P_{n+2} \pm K_{n+2}) \\ &\quad +k\,(P_{n+3} \pm K_{n+3})\ . \end{aligned} \tag{2.7}$$

Multiplication of the dual Pell quaternions is defined by

$$\begin{aligned} D_n^{P_1}\ \ D_n^{P_2} = &\ (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &\ (K_n + i\,K_{n+1} + j\,K_{n+2} + k\,K_{n+3}) \\ = &\ (P_n\,K_n) + i(\,P_nK_{n+1} + P_{n+1}K_n) + j\,(\,P_nK_{n+2} + P_{n+2}\,K_n) \\ &\ + k(\,P_nK_{n+3} + P_{n+3}\,K_n) \\ = &\ S_{D_n^{P_1}}\,S_{D_n^{P_2}} + \ S_{D_n^{P_1}}\,V_{D_n^{P_2}} \ + S_{D_n^{P_2}}\,V_{D_n^{P_1}}\ . \end{aligned} \tag{2.8}$$

The scaler and the vector part of D_n^P which is the n-th term of the dual Pell quaternion $(D_n^P\ )$ are denoted by

$$S_{D_n^P} = P_n \ \text{ and } \ V_{D_n^P} = i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}. \tag{2.9}$$

Thus, the dual Pell quaternion D_n^P is given by $D_n^P = S_{D_n^P} + V_{D_n^P}$.

Then, relation (2.8) is defined by

$$D_n^{P_1}\ D_n^{P_2} \ = S_{D_n^{P_1}}\ .\ S_{D_n^{P_2}} + \ S_{D_n^{P_1}}\ .\ V_{D_n^{P_2}} + S_{D_n^{P_2}}\ .\ V_{D_n^{P_1}}\ . \tag{2.10}$$

The conjugate of the dual Pell quaternion D_n^P is denoted by $\overline{D_n^P}$ and it is

$$\overline{D_n^P} \ = P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3}. \tag{2.11}$$

The norm of D_n^P is defined as

$$N_{D_n^P} = ||\,D_n^P\ ||^2 = \ D_n^P\ \ \overline{D_n^P} \ = \ P_n^2. \tag{2.12}$$

Then, we give the following theorem using statements (2.1), (2.2).

Theorem 2.1. Let P_n *and* D^P_n *be the n-th terms of the Pell sequence* (P_n) *and the dual Pell quaternion sequence* (D^P_n)*, respectively. In this case, for* $n \geq 1$ *we can give the following relations:*

$$D^P_n + \overline{D^P_n} = 2\,P_n, \tag{2.13}$$

$$(D^P_n)^2 = 2\,P_n\;D^P_n - P^2_n, \tag{2.14}$$

$$2\,D^P_{n+1} + D^P_n = D^P_{n+2}, \tag{2.15}$$

$$D^P_n - i\,D^P_{n+1} - j\,D^P_{n+2} - k\,D^P_{n+3} = P_n, \tag{2.16}$$

$$D^P_n\,D^P_m + D^P_{n+1}\,D^P_{m+1} = 2\,D^P_{n+m+1} - P_{n+m+1}. \tag{2.17}$$

Proof. Proof of first three equality can easily be done by the equations

$$D^P_n = P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3} \tag{2.18}$$

and

$$D^P_{n+1} = P_{n+1} + i\,P_{n+2} + j\,P_{n+3} + k\,P_{n+4}. \tag{2.19}$$

(2.13):

$$\begin{aligned} D^P_n + \overline{D^P_n} &= (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &\quad + (P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3}) \\ &= (P_n + P_n) + i(P_{n+1} - P_{n+1}) + j(P_{n+2} - P_{n+2}) \\ &\quad + k(P_{n+3} - P_{n+3}) \\ &= 2P_n. \end{aligned} \tag{2.20}$$

(2.14):

$$\begin{aligned} (D^P_n)^2 &= (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &\quad (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &= (P_n\,P_n) + i(P_nP_{n+1} + P_{n+1}P_n) + j(P_nP_{n+2} + P_{n+2}P_n) \\ &\quad + k(P_nPn{+}3 + Pn{+}3P_n) \\ &= P_n\,P_n + 2i\,P_n\,P_{n+1} + 2j\,P_n\,P_{n+2} + 2k\,2\,P_n\,P_{n+3} \\ &= 2P_n(P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) - P^2_n \\ &= 2P_n\,D^P_n - P^2_n. \end{aligned} \tag{2.21}$$

(2.15):

$$\begin{aligned} 2D^P_{n+1} + D^P_n &= 2\,(P_{n+1} + i\,P_{n+2} + j\,P_{n+3} + k\,P_{n+4}) \\ &\quad + (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &= (2\,P_{n+1} + P_n) + i\,(2P_{n+2} + P_{n+1}) + j\,(2P_{n+3} + P_{n+2}) \\ &\quad + k\,(2P_{n+4} + P_{n+3}) \\ &= P_{n+2} + i\,P_{n+3} + j\,P_{n+4} + k\,P_{n+5} \\ &= D^P_{n+2}\,. \end{aligned} \tag{2.22}$$

(2.16):

$$\begin{aligned} D^P_n - i\,D^P_{n+1} - j\,D^P_{n+2} - k\,D^P_{n+3} &= (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &\quad - i(P_{n+1} + i\,P_{n+2} + j\,P_{n+3} + k\,P_{n+4}) \\ &\quad - j(P_{n+2} + i\,Pn{+}3 + j\,Pn{+}4 + kPn{+}5) \\ &\quad - k(P_{n+3} + i\,P_{n+4} + j\,P_{n+5} + k\,P_{n+6}) \\ &= P_n\,. \end{aligned} \tag{2.23}$$

(2.17):

$$D_n^P D_m^P = P_n P_m + i(P_n P_{m+1} + P_{n+1} P_m) + j(P_n P_{m+2} + P_{n+2} P_m) + k(P_n P_{m+3} + P_{n+3} P_m) \tag{2.24}$$

and

$$\begin{aligned} D_{n+1}^P D_{m+1}^P = {} & (P_{n+1} P_{m+1}) + i(P_{n+1} P_{m+2} + P_{n+2} P_{m+1}) \\ & + j(P_{n+1} P_{m+3} + P_{n+3} P_{m+1}) \\ & + k(P_{n+1} P_{m+4} + P_{n+4} P_{m+1}). \end{aligned} \tag{2.25}$$

Finally, adding equations (2.24) and (2.25) side by side, we obtain

$$\begin{aligned} D_n^P D_m^P + D_{n+1}^P D_{m+1}^P &= P_{n+m+1} + 2i(P_{n+m+2}) \\ &\quad + 2j(P_{n+m+3}) + 2k(P_{n+m+4})] \\ &= 2D_{n+m+1}^P - P_{n+m+1} \end{aligned} \tag{2.26}$$

where we used following relations:

$$\begin{cases} P_n P_{n+1} + P_{n-1} P_n = P_{2n}, \\ P_n P_m + P_{n+1} P_{m+1} = P_{n+m+1}. \end{cases} \tag{2.27}$$

□

Theorem 2.2. Let D_n^P and D_n^p be the n-th terms of the dual Pell quaternion sequence (D_n^P) and the dual Pell-Lucas quaternion sequence (D_n^p), respectively. The following relations are satisfied

$$\begin{aligned} & D_{n+1}^P + D_{n-1}^P = D_n^p, \\ & D_n^P + D_{n+1}^P = \tfrac{1}{2} D_{n+1}^p, \\ & D_{n+2}^P - D_{n-2}^P = 2D_n^p. \end{aligned} \tag{2.28}$$

Proof. From equations (2.18), (2.19), it follows that

$$\begin{aligned} D_{n+1}^P + D_{n-1}^P &= (P_{n+1} + iP_{n+2} + jP_{n+3} + kP_{n+4}) \\ &\quad + (P_{n-1} + iP_n + jP_{n+1} + kP_{n+2}) \\ &= (P_{n+1} + P_{n-1}) + i(P_{n+2} + P_n) \\ &\quad + j(P_{n+3} + P_{n+1}) + k(P_{n+4} + P_{n+2}) \\ &= q_n + iq_{n+1} + jq_{n+2} + kq_{n+3} \\ &= D_n^p, \end{aligned} \tag{2.29}$$

$$\begin{aligned} D_n^P + D_{n-1}^P &= (P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}) \\ &\quad + (P_{n+1} + iP_{n+2} + jP_{n+3} + kP_{n+4}) \\ &= (P_n + P_{n+1}) + i(P_{n+1} + P_{n+2}) \\ &\quad + j(P_{n+2} + P_{n+3}) + k(P_{n+3} + P_{n+4}) \\ &= \tfrac{1}{2}[q_{n+1} + iq_{n+2} + jq_{n+3} + kq_{n+4} \\ &= \tfrac{1}{2} D_{n+1}^p, \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} D_{n+2}^P - D_{n-2}^P &= (P_{n+2} + iP_{n+3} + jP_{n+4} + kP_{n+5}) \\ &\quad - (P_{n-2} + iP_{n-1} + jP_n + kP_{n+1}) \\ &= (P_{n+2} - P_{n-2}) + i(P_{n+3} - P_{n-1}) \\ &\quad + j(P_{n+4} - P_n) + k(P_{n+5} - P_{n+1}) \\ &= 2q_n + i\,2q_{n+1} + j\,2q_{n+2} + k\,2q_{n+3} \end{aligned} \tag{2.31}$$

$$= 2D_n^p$$

where we used relations following:

$$\begin{cases} P_{n+1} + P_{n-1} = q_n, \\ P_n + P_{n+1} = \frac{1}{2} qn+1, \\ P_{n+2} - P_{n-2} = 2\, q_n. \end{cases} \tag{2.32}$$

□

Theorem 2.3. Let D_n^P *be the n-th term of the dual Pell quaternion sequence* (D_n^P) *and* $\overline{D_n^P}$ *be conjugate of* D_n^P *. Then, we can give the following relations between these quaternions:*

$$\begin{aligned} (D_n^P)^2 &= 2\,P_n\,D_n^P - P_n^2, \\ (D_n^P)^2 + (D_{n-1}^P)^2 &= 2D_{2n-1}^P - P_{2n-1}, \\ D_n^P\,\overline{D_n^P} + D_{n-1}^P\,\overline{D_{n-1}^P} &= P_n^2 + P_{n-1}^2 = P_{2n-1}, \\ D_{n+1}^P\,\overline{D_{n+1}^P} + D_n^P\,\overline{D_n^P} &= P_{n+1}^2 + P_n^2 = P_{2n+1}, \\ D_{n+1}^P\,\overline{D_{n+1}^P} - D_{n-1}^P\,\overline{D_{n-1}^P} &= P_{n+1}^2 - P_{n-1}^2 = 2\,P_{2n}. \end{aligned} \tag{2.33}$$

Proof. It can be proved easily by using (2.11) and (2.12). Now, we will prove first two equalities

$$\begin{aligned} (D_n^P)^2 &= P_n P_n + i\,(P_n P_{n+1} + P_{n+1} P_n) + j\,(P_n P_{n+2} + P_{n+2} P_n) \\ &\quad + k\,(P_n P_{n+3} + P_{n+3}\,P_n) \\ &= 2\,P_n(P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) - P_n^2 \\ &= 2P_n\,D_n^P - P_n^2 \end{aligned} \tag{2.34}$$

and using this identity, we get

$$\begin{aligned} (D_n^P)^2 + (D_{n-1}^P)^2 &= 2\,P_n\,D_n^P - P_n^P - 2\,P_{n-1}\,D_{n-1}^P - P_{n-1}^2 \\ &= 2\,P_n(P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) - P_n^2 \\ &\quad + 2\,P_{n-1}(P_{n-1} + i\,P_n + j\,P_{n+1} + k\,P_{n+2}) - P_{n-1}^2 \\ &= 2\,(P_n^2 + P_{n-1}^2) + 2i\,(P_n P_{n+1} + P_{n-1} P_n) \\ &\quad + j\,(P_n P_{n+2} + P_{n-1} P_{n+1}) + k\,(P_n\,P_{n+3} + P_{n-1} P_{n+2}) - (P_n^2 + P_{n-1}^2) \\ &= 2\,(P_{2n-1} + i\,P_{2n} + j\,P_{2n+1} + k\,P_{2n+2}) - P_{2n-1} \\ &= 2D_{2n-1}^P - P_{2n-1}. \end{aligned} \tag{2.35}$$

We can prove last three equalities by using equations (1.7) and (2.12) as follows:

$$\begin{aligned} D_n^P\,\overline{D_n^P} + D_{n-1}^P\,\overline{D_{n-1}^P} &= P_n^2 + P_{n-1}^2 = P_{2n-1}, \\ D_{n+1}^P\,\overline{D_{n+1}^P} + D_n^P\,\overline{D_n^P} &= P_{n+1}^2 + P_n^2 = P_{2n+1}, \\ D_{n+1}^P\,\overline{D_{n+1}^P} - D_{n-1}^P\,\overline{D_{n-1}^P} &= P_{n+1}^2 - P_{n-1}^2 = 2\,P_{2n}. \end{aligned} \tag{2.36}$$

□

Theorem 2.4. Let D_n^P *be the n-th term of dual Pell quaternion sequence* (D_n^P)*. Then, we have the following identities*

$$\sum_{s=1}^{n} D_s^P = \frac{1}{4}\,[D_{n+1}^p - D_1^p], \tag{2.37}$$

$$\sum_{s=0}^{p} D_{n+s}^P = \frac{1}{4}\,[D_{n+p+1}^p - D_{n+1}^p], \tag{2.38}$$

$$\sum_{s=1}^{n} D^P_{2s\text{-}1} = \tfrac{1}{2}\,[D^P_{2n} - D^p_0], \tag{2.39}$$

$$\sum_{s=1}^{n} D^P_{2s} = \tfrac{1}{2}\,[D^P_{2n+1} - D^P_1]. \tag{2.40}$$

Proof. (2.37) Hence, we can write

$$\begin{aligned}\sum_{s=1}^{n} D^P_s &= \sum_{s=1}^{n} P_s + i\sum_{s=1}^{n} P_{s+1} + j\sum_{s=1}^{n} P_{s+2} + k\sum_{s=1}^{n} P_{s+3}\\
&= \tfrac{1}{2}\,[(P_n + P_{n+1} - P_1 - P_0) + i\,(P_{n+1} + P_{n+2} - P_2 - P_1)\\
&\quad + j\,(P_{n+2} + P_{n+3} - P_3 - P_2) + k\,(P_{n+3} + P_{n+4} - P_4 - P_3)]\\
&= \tfrac{1}{2}\,[D^P_n + D^P_{n+1} - D^P_1 - D^P_0]\\
&= \tfrac{1}{2}\,[D^P_n + D^P_{n+1} - \tfrac{1}{2}D^p_1]\\
&= \tfrac{1}{4}\,[D^p_{n+1} - D^p_1].\end{aligned} \tag{2.41}$$

(2.38): Hence, we can write

$$\begin{aligned}\sum_{s=0}^{p} D^P_{n+s} &= \sum_{s=0}^{p} P_{n+s} = i\sum_{s=0}^{p} P_{n+s+1} + j\sum_{s=0}^{p} P_{n+s+2} + k\sum_{s=0}^{p} P_{n+s+3}\\
&= \tfrac{1}{2}\,[(P_{n+p+1} + P_{n+p} - P_{n+1} - P_n)\\
&\quad + i\,(P_{n+p+2} + P_{n+p+1} - P_{n+2} - P_{n+1})\\
&\quad + j\,(P_{n+p+3} + P_{n+p+2} - P_{n+3} - P_{n+2})\\
&\quad + k\,(P_{n+p+4} + P_{n+p+3} - P_{n+4} - P_{n+3})]\\
&= \tfrac{1}{2}\,[D^P_{n+p+1} + D^P_{n+p} - D^P_{n+1} - D^P_n]\\
&= \tfrac{1}{4}\,[D^p_{n+p+1} - D^p_{n+1}].\end{aligned} \tag{2.42}$$

(2.39) Hence, we can write

$$\begin{aligned}\sum_{s=1}^{n} D^P_{2s\text{-}1} &= \sum_{s=1}^{n} P_{2s\text{-}1} + i\sum_{s=1}^{n} P_{2s} + j\sum_{s=1}^{n} P_{2s+1} + k\sum_{s=1}^{n} P_{2s+2}\\
&= (P_1 + P_3 + \ldots + P_{2n-1}) + i(P_2 + P_4 + \ldots + P_{2n})\\
&\quad + j(P_3 + P_5 + \ldots + P_{2n+1}) + k(P_4 + P_6 + \ldots + P_{2n+2})\\
&= \tfrac{1}{2}\,[(P_{2n} - P_0) + i\,(P_{2n+1} - P_1) + j\,(P_{2n+2} - P_2) + k\,(P_{2n+3} - P_3)]\\
&= \tfrac{1}{2}\,[P_{2n} + i\,P_{2n+1} + j\,P_{2n+2} + k\,P_{2n+3}] - \tfrac{1}{2}\,[P_0 + i\,P_1 + j\,P_2 + k\,P_3]\\
&= \tfrac{1}{2}\,[D^P_{2n} - D^P_0].\end{aligned} \tag{2.43}$$

(2.40): Hence, we obtain

$$\sum_{s=1}^{n} D^P_{2s} = (P_2 + P_4 + \ldots + P_{2n}) + i\,(P_3 + P_5 + \ldots + P_{2n+1})$$

$$
\begin{aligned}
&+ j(P_4 + P_6 + \ldots + P_{2n+2}) + k(P_5 + P_7 + \ldots + P_{2n+3})\\
&= \tfrac{1}{2}\,[(P_{2n+1} - P_1) + i\,(P_{2n+2} - P_2) + j\,(P_{2n+3} - P_3)\\
&\quad + k(P_{2n+4} - P_4)] \qquad (2.44)\\
&= \tfrac{1}{2}\,[\,P_{2n+1} + i\,P_{2n+2} + j\,P_{2n+3} + k\,P_{2n+4}]\\
&\quad - \tfrac{1}{2}\,[\,P_1 + i\,P_2 + j\,P_3 + k\,P_4]\\
&= \tfrac{1}{2}\,[D^P_{2n+1} - D^P_1]\,.
\end{aligned}
$$

□

Theorem 2.5. Let D^P_n and D^p_n be the n-th terms of the dual Pell quaternion sequence (D^p_n) and the dual Pell-Lucas quaternion sequence (D^p_n) , respectively. Then, we have

$$D^p_n\,\overline{D^P_n} - \overline{D^p_n}\,D^P_n = 2[P_n\,D^p_n - q_n\,D^P_n], \qquad (2.45)$$

$$D^p_n\,\overline{D^P_n} - \overline{D^p_n}\,D^P_n = 2\,q_n.P_n = 2\,P_{2n}, \qquad (2.46)$$

$$D^p_n D^P_n - \overline{D^p_n}\;\overline{D^P_n} = 2\,[P_n\,D^p_n + q_n\,D^P_n - 2\,P_{2n}]\,, \qquad (2.47)$$

$$D^p_n D^P_n + \overline{D^p_n}\;\overline{D^P_n} = 2\,P_{2n}. \qquad (2.48)$$

Proof. (2.45):

$$
\begin{aligned}
D^p_n\,\overline{D^P_n}\;\overline{D^p_n} &= (q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3})\\
&\quad (P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3})\\
&\quad -(q_n - i\,q_{n+1} - j\,q_{n+2} - k\,q_{n+3})\\
&\quad (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3})\\
&= (q_n P_n - q_n P_n)\\
&\quad +2i\,(q_{n+1} P_n - q_n P_{n+1})\\
&\quad +2j\,(q_{n+2} P_n - q_n P_{n+2})\\
&\quad +2k\,(q_{n+3} P_n - q_n P_{n+3})\\
&= 2P_n(q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3})\\
&\quad -2q_n(P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3})\\
&= 2[P_n\,D^p_n - q_n\,D^P_n].
\end{aligned} \qquad (2.49)
$$

(2.46):

$$
\begin{aligned}
D^p_n\,\overline{D^P_n} + \overline{D^p_n}\,D^P_n &= (q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3})\\
&\quad (P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3})\\
&\quad +(q_n - i\,q_{n+1} - j\,q_{n+2} - k\,q_{n+3})\\
&\quad (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3})\\
&= q_n[P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3}]\\
&\quad +(i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3})P_n\\
&\quad +q_n[P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}]
\end{aligned} \qquad (2.50)
$$

$$+(-i\, q_{n+1} - j\, q_{n+2} - k\, q_{n+3})P_n$$
$$= q_n[\, \overline{D_n^P} + D_n^P\,]$$
$$= 2\, P_{2n}.$$

where we used relation (2.13).

(2.47):

$$\begin{aligned} D_n^p\, D_n^P - \overline{D_n^p}\, D_n^P &= (q_n + i\, q_{n+1} + j\, q_{n+2} + k\, q_{n+3}) \\ &\quad (P_n + i\, P_{n+1} + j\, P_{n+2} + k\, P_{n+3}) \\ &\quad -(q_n - i\, q_{n+1} - j\, q_{n+2} - k\, q_{n+3}) \\ &\quad (P_n - i\, P_{n+1} - j\, P_{n+2} - k\, P_{n+3}) \\ &= 2(q_nP_n - q_nP_n) + 2i\,(q_nP_{n+1} + q_{n+1}P_n) \\ &\quad +2j\,(q_n\, P_{n+2} + q_{n+2}\, P_n) + 2k\,(q_n\, P_{n+3} + q_{n+3}P_n) \\ &= 2q_n\, D_n^P + 2\, P_n\, D_n^p - 4\, P_{2n}) \\ &= 2\,[P_n\, D_n^p + q_n\, D_n^P - 2\, P_{2n}]. \end{aligned} \tag{2.51}$$

(2.48):

$$\begin{aligned} D_n^p\, D_n^P + \overline{D_n^p}\;\overline{D_n^P} &= (q_n + i\, q_{n+1} + j\, q_{n+2} + k\, q_{n+3}) \\ &\quad (P_n + i\, P_{n+1} + j\, P_{n+2} + k\, P_{n+3}) \\ &\quad +(q_n - i\, q_{n+1} - j\, q_{n+2} - k\, q_{n+3}) \\ &\quad (P_n - i\, P_{n+1} - j\, P_{n+2} - k\, P_{n+3}) \\ &= q_n[P_n + i\,(P_{n+1} - P_{n+1}) \\ &\quad +j\,(P_{n+2} - P_{n+2}) + k(P_{n+3} - P_{n+3})] \\ &\quad +P_n[q_n + i\,(q_{n+1} - q_{n+1}) \\ &\quad +j\,(q_{n+2} - q_{n+2}) + k\,(q_{n+3} - q_{n+3})] \\ &= 2q_nP_n = 2P_{2n}. \end{aligned} \tag{2.52}$$

□

Theorem 2.6. (Binet's Formulas). Let D_n^P *and* D_n^p *be n – th terms of dual Pell quaternion sequence* (D_n^P) *and the dual Pell-Lucas quaternion sequence* (D_n^p)*, respectively. For* $n \geq 1$*, the Binet's formulas for these quaternions are as follows:*

$$D_n^P = \frac{1}{\alpha - \beta}[\underline{\alpha}\, \alpha^n - \underline{\beta}\, \beta^n] \tag{2.53}$$

and

$$D_n^P = [\underline{\underline{\alpha}}\, \alpha^n - \underline{\underline{\beta}}\, \beta^n] \tag{2.54}$$

respectively, where

$$\underline{\alpha} = 1 + i\,(2 - \beta) + j\,(5 - 2\beta) + k\,(12 - 5\beta),\ \alpha = 1 + \sqrt{2}\,,$$

$$\underline{\beta} = -1 + i\,(\alpha - 2) + j\,(2\alpha - 5) + k\,(5\alpha - 12),\ \beta = 1 - \sqrt{2}$$

and

$$\underline{\underline{\alpha}} = (2 - 2\beta) + i\,(6 - 2\beta) + j\,(14 - 6\beta) + k\,(34 - 14\beta),\ \alpha = 1 + \sqrt{2}\,,$$

$$\underline{\underline{\beta}} = (2\alpha - 2) + i\,(2\alpha - 6) + j\,(6\alpha - 14) + k\,(14\alpha - 34),\ \beta = 1 - \sqrt{2}$$

respectively.

Proof. The characteristic equation of recurrence relation $D^P_{n+2} = 2\,D^P_{n+1} + D^P_n$ is

$$t^2 - 2t - 1 = 0. \tag{2.55}$$

The roots of this equation are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$

where $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$, $\alpha\beta = -1$.

Using recurrence relation and initial values $D^P_0 = (0, 1, 2, 5)$,
$D^P_1 = (1, 2, 5, 12)$ the Binet's formula for D^P_n, we get

$$D^P_n = A\,\alpha^n + B\,\beta^n = \frac{1}{3}\,[\underline{\alpha}\,\alpha^n - \underline{\beta}\,\beta^n]\,, \tag{2.56}$$

where $A = \dfrac{D^P_1 - D^P_0\,\beta}{\alpha - \beta}$, $B = \dfrac{\alpha\,D^P_0 - D^P_1}{\alpha - \beta}$ and

$\underline{\alpha} = 1 + i\,(2 - \beta) + j\,(5 - 2\beta) + k\,(12 - 5\beta)$,

$\underline{\beta} = -1 + i\,(\alpha - 2) + j\,(2\alpha - 5) + k\,(5\alpha - 12)$. □

Similarly, using recurrence relation $D^p_{n+2} = D^p_{n+1} + 2\,D^p_n$, the Binet's formula for D^p_n is obtained as follows:

$$D^p_n = (\underline{\underline{\alpha}}\,\alpha^n - \underline{\underline{\beta}}\,\beta^n)\,. \tag{2.57}$$

Theorem 2.7. (Cassini-like Identity). Let D^P_n and D^p_n be n –th terms of dual Pell quaternion sequence (D^P_n) and the dual Pell-Lucas quaternion sequence (D^p_n) are as follows:

$$D^P_{n-1}\,D^P_{n+1} - (D^P_n)^2 = (-1)^n\,(1 + 2i + 6j + 14k), \tag{2.58}$$

and

$$D^p_{n-1}\,D^p_{n+1} - (D^p_n)^2 = 8\,(-1)^{n+1}\,(1 + 2i + 6j + 14k) \tag{2.59}$$

respectively.

Proof. (2.58):

$$\begin{aligned} D^P_{n-1}\,D^P_{n+1} - (D^P_n)^2 &= (P_{n-1} + i\,P_n + j\,P_{n+1} + k\,P_{n+2}) \\ &\quad (P_{n+1} + i\,P_{n+2} + j\,P_{n+3} + k\,P_{n+4}) \\ &\quad -(P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &\quad (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &= (P_{n-1}P_{n+1} - P^2_n) \\ &\quad +i\,(P_{n-1}\,P_{n+2} - P_n\,P_{n+1}) \\ &\quad +j\,(P_{n-1}P_{n+3} - 2P_nP_{n+2} + P^2_{n+1}) \\ &\quad +k\,(P_{n-1}\,P_{n+4} + P_{n+1}\,P_{n+2}) - 2P_nP_{n+3}) \\ &= (-1)^n\,(1 + 2i + 6j + 14k). \end{aligned} \tag{2.60}$$

and
(2.59):

$$\begin{aligned} D^p_{n-1}\,D^p_{n+1} - (D^p_n)^2 &= (q_{n-1} + i\,q_n + j\,q_{n+1} + k\,q_{n+2}) \\ &\quad (q_{n+1} + i\,q_{n+2} + j\,q_{n+3} + k\,q_{n+4}) \\ &\quad -(q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}) \\ &\quad (q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}) \\ &= (q_{n-1}q_{n+1} - q^2_n) \end{aligned} \tag{2.61}$$

$$+ i\,(q_{n-1}q_{n+2} - q_n q_{n+1})$$
$$+ j\,(q_{n-1}q_{n+3} - 2q_n q_{n+2} + q_{n+1}^2)$$
$$+ k\,(q_{n-1}q_{n+4} + q_{n+1}q_{n+2} - 2q_n q_{n+3})$$
$$= 8\,(-1)^{n+1}\,(1 + 2i + 6j + 14k).$$

where we use identities of Pell numbers and Pell-Lucas numbers as follows:

$$P_m P_{n-1} - P_{m-1} P_n = (-1)^n P_{m-n},\ P_{n+2} = 2P_{n+1} + P_n,$$
$$q_m q_{n-1} - q_{m-1} q_n = (-1)^{n+1} q_{m-n},\ q_{n+2} = q_{n+1} + 2q_n. \tag{2.62}$$

We will give an example in which we check in a particular case the Cassini-like identity for dual Pell quaternions and for the dual Pell-Lucas quaternions. □

Example 1. Let D_1^P, D_2^P, D_3^P and D_4^P be the dual Pell quaternions such that

$$D_1^P = 1 + 2i + 5j + 12k,$$
$$D_2^P = 2 + 5i + 12j + 29k,$$
$$D_3^P = 5 + 12i + 29j + 70k,$$
$$D_4^P = 12 + 29i + 70j + 169k.$$

In this case,

$$\begin{aligned} D_1^P D_3^P - (D_2^P)^2 &= (1 + 2i + 5j + 12k)\,(5 + 12i + 29j + 70k) \\ &\quad -(2 + 5i + 12j + 29k)2 \\ &= (5 + 22i + 54j + 130k) - (4 + 20i + 48j + 116k) \\ &= (-1)^2\,(1 + 2i + 6j + 14k) \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} D_2^P D_4^P - (D_3^P)^2 &= (2 + 5i + 12j + 29k)\,(12 + 29i + 70j + 169k) \\ &\quad -(5 + 12i + 29j + 70k)^2 \\ &= (24 + 118i + 284j + 686k) \\ &\quad -(25 + 120i + 290j + 700k) \\ &= (-1 - 2i - 6j - 14k) \\ &= (-1)3\,(1 + 2i + 6j + 14k). \end{aligned} \tag{2.64}$$

Example 2. Let D_1^p, D_2^p, D_3^p and D_4^p be the dual Pell-Lucas quaternions such that

$$D_1^p = 2 + 6i + 14j + 34k,$$
$$D_2^p = 6 + 14i + 34j + 82k,$$
$$D_3^p = 14 + 34i + 82j + 198k,$$
$$D_4^p = 34 + 82i + 198j + 478k.$$

In this case,

$$\begin{aligned} D_1^p D_3^p - (D_1^p)^2 &= (2 + 6i + 14j + 34k)\,(14 + 34i + 82j + 198) \\ &\quad -(6 + 14i + 34j + 82k)2 \\ &= (28 + 152i + 360j + 872k) \\ &\quad -(36 + 168i + 408j + 984k) \\ &= -(8 + 16i + 48j + 112k) \\ &= 8\,(-1)^3\,(1 + 2i + 6j + 14k) \end{aligned} \tag{2.65}$$

and

$$\begin{aligned} D_2^p D_4^p - (D_3^p)^2 &= (6 + 14i + 34j + 82k)\,(34 + 82i + 198j + 478k) \\ &\quad -(14 + 34i + 82j + 198k)^2 \\ &= (204 + 968i + 2344j + 5656k) \end{aligned}$$

$$
\begin{aligned}
& -(196 + 952i + 2296j + 5544k) \\
&= (8 + 16i + 48j + 112k) \\
&= 8\,(-1)^4\,(1 + 2i + 6j + 14k). \qquad (2.66)
\end{aligned}
$$

3. **Conclusion**

The dual Pell quaternions are given by

$$D_n^P = P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3} \qquad (3.1)$$

where P_n is the n–th Pell number and i, j, k are quaternionic units which satisfy the equalities
$i^2 = j^2 = k^2 = i\,j\,k = 0,\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$

References

1. Hamilton W. R., Elements of Quaternions. Longmans, Green and Co., London, (1866)
2. Majernik, V., Quaternion formulation of the Galilean space-time transformation. Acta Phy. Slovaca. *56* (1), 9-14 (2006).
3. Ata, E., Yayl1, Y., Dual quaternions and dual projective spaces. Chaos Solitons Fractals. *40 (3)*, 1255-1263 (2009).
4. Nurkan K. S., and Güven A. İ., Dual Fibonacci Quaternions. Adv. Appl. Clifford Algebras. doi:10.1007/s00006-014-0488-7 (2014).
5. Ercan Z., Yüce, S., On properties of the dual quaternions. Eur. *J. Pure Appl. Math. 4 (2),* 142-146 (2011).
6. Yüce, S., Torunbalci Aydin, F., A New Aspect of Dual Fibonacci Quaternions. Adv. Appl. Clifford Algebras. *26 (2)*, 873-884 (2016).
7. Horadam A. F., Pell Identities. The Fibonacci Quarterly *9 (3)*, 245-263 (1971).
8. Horadam A. F. and Bro. J. M. Mahon., Pell and Pell-Lucas Polynomials. The Fibonacci Quarterly *23 (1)*, 17-20 (1985).
9. Horadam A.F., Quaternion Recurrence Relations. Ulam Quarterly. *2 (2)*, 23-33 (1993).
10. Çimen Cennet Bolat and İpek Ahmet., On Pell Quaternions and Pell-Lucas Quaternions. Adv. Appl. Clifford Algebras. *26,* 39-51 (2016).
11. Anetta Szynal-Liana and Iwona Wloch., The Pell Quaternions and the Pell Octanions. Adv. Appl. Clifford Algebras. *26,* 435-440 (2016).
12. Clifford, W. K., Preliminary Sketch of Bi-quaternions. Proc. London Math. Soc. *4,* 381-395 (1873).
13. Horadam A. F., Complex Fibonacci Numbers and Fibonacci Quaternions. American Math. Monthly. *70 (3),* 289-291 (1963).
14. Iyer M. R., A Note on Fibonacci Quaternions. The Fibonacci Quarterly. *7 (3)*, 225-229 (1969).
15. Iyer M. R., Some Results on Fibonacci Quaternions. The Fibonacci Quarterly. *7,* 201-210 (1969).
16. Swamy M.N., On Generalized Fibonacci Quaternions. The Fibonacci Quarterly, *11 (5),* 547-550 (1973).
17. Iakin A.L., Generalized Quaternions of Higher Order. The Fibonacci Quarterly. *15 (4),* 343-346 (1977).
18. Iakin A. L., Generalized Quaternions with Quaternion Components. The Fibonacci Quarterly. *15*, 350-352 (1977).
19. Kula L., Yayl1 Y., Split Quaternions and Rotations in Semi-Euclidean Space. J. Korean Math. Soc. *44 (6),* 1313-1327 (2007).
20. Hal1c1 S., On Fibonacci Quaternions. Adv. Appl. Clifford Algebras. *22 (2),* 321-327 (2012).
21. Hal1c1 S., On Complex Fibonacci Quaternions. Adv. Appl. Clifford Algebras. *23,* 105-112 (2013).
22. Akyiğit M., Kösal H. H. and Tosun M., Split Fibonacci Quaternions. Adv. Appl. Clifford Algebras. *23 (3),* 535-545 (2013).