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Method of Construction of Group Divisible Designs

C.P.S. CHOUHAN[1], M.S. DULAWAT[2] and J.S. PARIHAR[3]

[1]Department of Mathematics & Statistics, Bhupal Nobles' University, Udaipur, India
[2]Department of Mathematics & Statistics, Mohanlal Sukhadia University, Udaipur, India
[3]94, D.K. Devsthali, Bhopal, India
Email of Corresponding Author: dulawat_ms@rediffmail.com& chandrapal.singh1982@gmail.com
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Abstract

In this paper, we have presented a new method of construction of regular group divisible design. We also characterized a particular design of the series $v = b = 4t + 3,\ r = k = 2t + 1,\ \lambda = t$ viz., $v = b = 31,\ r = k = 15,\ \lambda = 7$, considering nice block structure.

Keywords: BIB design; GD design.

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1. Introduction

In this paper, the incidence matrix of the partially balanced incomplete block (PBIB) designs constructed are newly patterned by making use of incidence matrices of balanced incomplete block(BIB) designs. This is a unique type arrangement of incidence matrices of BIB designs, which further cannot be generalized. It gives a new series of regular group divisible (GD) designs. Some new regular GD designs are obtained.

We have also studied the block structure of symmetrical BIB design with parameters $v = b = 31,\ r = k = 15,\ \lambda = 7$, here we present two non-isomorphic solutions of this design. From solution-1, three different types of designs are obtained viz., semiregular, singular GD designs and symmetrical BIB designs with parameters $v = b = 15,\ r = k = 7,\ \lambda = 3$. The authors are unable to search any other design belonging to the series $v = b = 4t + 3,\ r = k = 2t + 1,\ \lambda = t$, for $t > 3$ by using the method of Bose, Shrikhande and Bhattacharya[1].

For some notations, N is the incidence matrix of order $v \times b$ of a BIB design, $\overline{N} = J - N$, $J_{r\times s}$ denotes a flat matrix of order $r \times s$ having all elements unity, $O_{p\times q}$ denotes matrix having all elements zero. I_n isan identity matrix of order n.

For complete descriptions and combinational properties of BIB designs and GD designs, we refer to Raghavarao[6].

2. Construction of GD designs:

Theorem 2.1: The existence of BIBD (v, b, r, k, λ) having incidence matrices N and $\overline{N}$ implies the existence of GD design with parameters:

$$v^* = 7v,\ b^* = 7b,\ r^* = b + 2r,\ k^* = v + 2k,\ \lambda_1^* = b - 2r + 4\lambda,\ \lambda_2^* = r;\ m = 7,\ n = v \tag{2.1}$$

having incidence matrix S as follows:

$$S = \begin{bmatrix} \overline{N} & N & N & O & N & O & O \\ O & \overline{N} & N & N & O & N & O \\ O & O & \overline{N} & N & N & O & N \\ N & O & O & \overline{N} & N & N & O \\ O & N & O & O & \overline{N} & N & N \\ N & O & N & O & O & \overline{N} & N \\ N & N & O & N & O & O & \overline{N} \end{bmatrix} \tag{2.2}$$

Proof: Let S be the incidence matrix of a GD design having row (column) sum equal to $r^* \times k^*$ of order $v^* \times b^*$ and S' be the transpose of S, then, we have

$$SS' = I_7 \otimes C + (J_7 - I_7) \otimes D \tag{2.3}$$

where

$$\begin{aligned} C &= 3NN' + \overline{N}\,\overline{N}' \\ &= 4(r - \lambda)I_V + (b - 2r + 4\lambda)J_V \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} D &= NN' + N\overline{N}' \\ &= rJ_V \end{aligned} \tag{2.5}$$

From the relations (2.4) and (2.5), we have $\lambda_1^* = b - 2r + 4\lambda$ and $\lambda_2^* = r$ respectively. The other parameters are obvious.

Similarly,

$$S'S = I_7 \otimes A + (J_7 - I_7) \otimes B \tag{2.6}$$

where

$$\begin{aligned} A &= 3N'N + \overline{N}'\,\overline{N} \\ &= 4N'N + (v - 2k)J_V \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} B &= N'\overline{N} + N'N \\ &= kJ_V \end{aligned} \tag{2.8}$$

Therefore, S is the incidence matrix of a GD design with parameters given as in (2.1). Since $r^*k^* > v^*\lambda_2^*$, therefore, the GD design so constructed is a regular GD design. Hence the theorem is established.

Remark 2.1: The form of S'S calculated above suggests that dual of the design constructed in Theorem 2.1 is also a GD design if N'N is a linear metric polynomial in I and J that is always possible if we start with a symmetric BIB design and in that case, we get a symmetric GD design.

In particular, if we start with a BIB design having v = 2k, see Preece[5], then the parameters of GD design given in (2.1) reduce to:

$$v^* = 7v,\ b^* = 7b,\ r^* = 4r,\ k^* = 4k,\ \lambda_1^* = 4\lambda,\ \lambda_2^* = r;\ m = 7,\ n = v \tag{2.9}$$

It is to be noted that the parameters of the GD design given in (2.9) follow the characterization property of regular GD design. Incidence structure S given in (2.2) cannot be generalized, because the constancy of C and D is not maintained in SS', consequently, $\lambda_1^* \text{ and } \lambda_2^*$ will not remain constant.

As an illustration, let us consider BIBD (4, 6, 3, 2, 1), we obtain regular GD design with parameters: $v^* = 28,\ b^* = 42,\ r^* = 12,\ k^* = 8,\ \lambda_1^* = 4,\ \lambda_2^* = 3;\ m = 7,\ n = 4$, which seems to be new design because it is not included in the list of Parihar[4], Dulawat *et al.*[2] also constructed new semi regular GD designs. On considering SBIBD (3, 3, 2, 2, 1), we obtain regular GD design with parameters:

$$v^* = 21,\ b^* = 21,\ r^* = 7,\ k^* = 7,\ \lambda_1^* = 3,\ \lambda_2^* = 2;\ m = 7,\ n = 3.$$

The same GD design is constructed by Freeman[3] as R180 a, using cyclic method, but our solution is non-isomorphic to the solution given by Freeman[3].

Remark 2.2: In the incidence matrix S defined in (2.2), if we replace $\overline{N}$ by N and N by J, then we obtain a new series of GD design with parameters: $v^* = 7v,\ b^* = 7b,\ r^* = 3b + r,\ k^* = 3v + k,\ \lambda_1^* = 3b + \lambda,\ \lambda_2^* = b + r;\ m = 7,\ n = v$

3. Block Structure Property of SBIBD (31, 31, 15, 15, 7):

In this section, we study a very interesting property of the SBIBD (31, 31, 15, 15, 7). Here, we present two non-isomorphic solutions of this design. Solution-1 is obtained on developing cyclically the initial block (1, 2, 5, 7, 8, 10, 11, 12, 13, 15, 17, 21, 24, 25, 26) mod 31. Solution-2 is obtained on developing cyclically the initial block (1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28) mod 31.

We mention the significant theorem by Bose, Shrikhande and Bhattacharya[1] which states as:

Theorem: By omitting the blocks in which a particular treatment say θ occurs from a BIB design $(v, b, r, k, \lambda = 1)$, we obtain a GD design with parameters:

$v^* = v - 1,\ b^* = b - r,\ r^* = r - 1,\ k^* = k,\ \lambda_1^* = 0,\ \lambda_2^* = 1;\ m = r,\ n = k - 1$, where two treatments belong to the same group, if they occur together in the same block as θ in the solution of the BIB design.

Our finding is significant from the viewpoint that SBIB design under consideration has $\lambda = 7$. We consider solution-1 of the SBIBD (31, 31, 15, 15, 7) by omitting a particular treatment say 31 and deleting all those blocks in which treatment number 31 is contained. The remaining blocks of the design give rise to a semi-regular GD design with parameters:

$$v' = 30,\ b' = 16,\ r' = 8,\ k' = 15,\ \lambda_1' = 0,\ \lambda_2' = 4;\ m = 15,\ n = 2 \tag{3.1}$$

The GD association scheme is as follows:

1	2	3	4	6	7	8	9	11	12	13	15	17	21	26
14	28	5	25	10	16	19	24	23	20	30	22	18	27	29

Further, on omitting treatment number 31 from the deleted blocks and hence forming the blocks with the remaining treatments from all the deleted blocks, we obtain singular GD design with parameters:

$$v^* = 30,\ b^* = 15,\ r^* = 7,\ k^* = 14,\ \lambda_1^* = 7,\ \lambda_2^* = 3;\ m = 15,\ n = 2 \tag{3.2}$$

Designs given in (3.1) and (3.2) are based on the same association scheme conversely, if each group of the GD association scheme of the singular GD design is replaced by a treatment, then, we obtain SBIBD with parameters:

$$v = b = 15,\ r = k = 7,\ \lambda = 3 \tag{3.3}$$

Our solution may be non-isomorphic to the known solution given in Raghavarao[6]. It is to be noted that from solution-2 of SBIBD (31, 31, 15, 15, 7) the designs mentioned in (3.1), (3.2) and (3.3) are not obtained and hence solution-1 & solution-2 of SBIBD (31, 31, 15, 15, 7) are non-isomorphic.

Further, using the same technique as in SBIBD (31, 31, 15, 15, 7) for the solution of SBIBD (15, 15, 7, 7, 3) obtained from singular GD design given in (3.2), we obtain the following designs:

(i) Semi-regular GD design with parameters:

$$v^* = 14,\ b^* = 8,\ r^* = 4,\ k^* = 7,\ \lambda_1^* = 0,\ \lambda_2^* = 2;\ m = 7,\ n = 2$$

(ii) Singular GD design with parameters:

$$v^* = 14,\ b^* = 7,\ r^* = 3,\ k^* = 6,\ \lambda_1^* = 3,\ \lambda_2^* = 1;\ m = 7,\ n = 2$$

(iii) Symmetrical BIBD with parameters:

$$v = b = 7,\ r = k = 3,\ \lambda = 1$$

The authors are unable to search other designs of the series of SBIBD $(v = b = 4t + 3,\ r = k = 2t + 1,\ \lambda = t)$ for $t > 1$ except SBIBD $(v = b = 15,\ r = k = 7,\ \lambda = 3)$, which has nice block structure property like SBIBD $(v = b = 31,\ r = k = 15,\ \lambda = 7)$.

Hence, we have conjecture:

Conjecture: There does not exist any other design belonging to the series of SBIBD $(v = b = 4t + 3,\ r = k = 2t + 1,\ \lambda = t)$ for $t > 1$ except SBIBD and SBIBD, which generates the three designs viz, semi-regular, singular GD designs and SBIBD by using this method.

References

1. Bose, R.C., Shrikhande, S.S. and Bhattacharya. K.N., *On the construction of group divisible incomplete block designs, Ann. Math. Statist. 24,* 167-195 (1953).
2. Dulawat, M.S., Solanki, A.S. and Parihar, J.S., *Construction of semi-regular GD designs, GanitaSandesh 24 (1),* 39-46 (2010).
3. Freeman, G.H., *A cyclic method of constructing regular group divisible incomplete block designs, Biometrika* 63, 3, 555-558 (1976).
4. Parihar, J.S., *Covering theorems and construction of incomplete block designs. Unpublished Ph.D. Thesis, University of Indore, Indore, India* (1981).
5. Preece, D.A., *Incomplete block designs with v = 2k, Sankhya A 29,* 305-316 (1967).
6. Raghavarao, D., *Constructions and Combinatorial Problems in Design of Experiments, John Wiley, New York* (1971).