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Characterizations of conjugate unitary matrices

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Abstract

The concept of conjugate unitary matrices is introduced. Characterizations of conjugate unitary matrices are obtained and derived some theorems.
Key words: unitary matrix, secondary transpose of a matrix, conjugate secondary transpose of a matrix, conjugate unitary matrix.
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1. Introduction

Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{nxn}$. Let $A^T, \overline{A}, A^*, A^S, A^\theta$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Let 'V' be the associated permutation matrix whose elements on the secondary diagonal are 1, other elements are zero. Also 'V' satisfies the following properties .

$V^T = \overline{V} = V^* = V, V^2 = I$ and $VV^* = V^*V = I$

A matrix $A \in C_{nxn}$ is called unitary if $AA^* = A^*A = I$. [8]

A matrix $A \in C_{nxn}$ is called conjugate normal if $AA^* = \overline{A^*A}$ [2]

In this paper the conjugate unitary matrix is defined and its characterizations are discussed. Characterizations of conjugate unitary matrices analogous to that of conjugate normal matrices are obtained and established some theorems. The conditions for sums and products of conjugate unitary matrices to be conjugate unitary are discussed.

1.1 Literature Review :

The concept of normal matrices over the complex field was introduced by Toeplitz who gave necessary and

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sufficient conditions for a complex matrix to be normal. A matrix A is normal if $AA^* = A^*A$ [1]. This concept of normal was introduced as a generalization of hermitian matrices. Sadkane[7] has studied about the normal matrices on singular valus. The concept of unitary matrix was introduced as a special case of normal matrix. Meenakshi. A.R. and Krishnamoorthy. S.[6] introduceed the concept of range k-hermitian matrices. The concept and characterizations of s-normal matrices is introduced by Krishnamoorthy. S. and Vijayakumar. R. Some equivalent conditions on s-normal matrices are also derived[5]. Krishnamoorthy. S. and Govindarasu. A introduceed the concept of Secondary Unitary Matrices[4].

The importance of normal matrices explains the appearance of the survey[3]. As putforth by Grone. R, Johnson. C.R., Sa. E.M., Wolkowicz. H, it is hoped that it will be useful to a wide audience, a long list of conditions on an nxn complex matrix A, equivalent to its being normal, is presented. Elsner. L and Ikramov. Kh.D. [1] have given a list of 70 conditions on an nxn complex matrix A equivalent to its being normal.

1.2 Research Methodology:

For a number of decades matrices over the real and complex fields have been intently by every beginning student of Linear Algebra. These are defined the property of being matrix about main diagonal. Anna Lee has initiated the study of secondary symmetric and skew symmetric matrices about the secondary diagonal.

A matrix A is normal if $AA^* = A^*A$ [1].

A matrix A ∈ C_{nxn} is called unitary if AA* = A*A = I. [5] A matrix A ∈ C_{nxn} is called conjugate normal if $AA^* = \overline{A^*A}$ [2].

Result and Discussion

2. Conjugate Unitary Matrix :

Definition.2.1 Let A ∈ C_{nxn}. A is said to be conjugate unitary matrix if $AA^* = \overline{A^*A} = I$.

Theorem 2.2. Let A ∈ C_{nxn}. If A is conjugate unitary matrix then A^T is conjugate unitary matrix.

Proof: A is conjugate unitary matrix ⇒ $AA^* = \overline{A^*A} = I$.

Taking transpose of the above equation we get

$$(AA^*)^T = (\overline{A^*A})^T = I^T$$

$$(A^*)^T A^T = \overline{(A^*A\ )^T} = I$$

$$(A^*)^T A^T = \overline{A^T (A^*)^T} = I$$

$$(A^T)^* A^T = \overline{A^T (A^T)^*} = I$$

Taking conjugate of the above equation we get

$$\overline{(A^T)^* A^T} = \overline{\overline{A^T (A^T)^*}} = \bar{I}$$

$$\overline{(A^T)^* A^T} = A^T (A^T)^* = I (\text{since } \bar{I} = I)$$

$$A^T (A^T)^* = \overline{(A^T)^* A^T} = I$$

∴ A^T is conjugate unitary matrix.

Theorem 2.3. Let A ∈ C_{nxn}. If A is conjugate unitary then A^* is conjugate unitary matrix.

Proof : A is conjugate unitary matrix ⇒ $AA^* = \overline{A^*A} = I$.

Taking conjugatetranspose of the above we get

$$(AA^*)^* = (\overline{A^*A})^* = I^*$$

$$(A^*)^* A^* = \overline{(A^*A\ )^*} = I \quad (\text{since } I^* = I)$$

$$(A^*)^* A^* = \overline{A^* (A^*)^*} = I$$

Taking conjugate of the above equations we get

$\overline{(A^*)^*A^*} = \overline{\overline{A^*(A^*)^*}} = \bar{I}$

$\overline{(A^*)^*A^*} = A^*(A^*)^* = I$ (since $\bar{I} = I$)

$A^*(A^*)^* = \overline{(A^*)^*A^*} = I$

$\therefore A^*$ is conjugate unitary matrix.

Theorem 2.4: Let A $\in$ C_{nxn}. If A is conjugate unitary matrix then A^{-1} is conjugate unitary matrix.

Proof : A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

Taking inverse of the above equation

$(AA^*)^{-1} = (\overline{A^*A})^{-1} = I^{-1}$

$(A^*)^{-1}A^{-1} = \overline{(A^*A)^{-1}} = I^{-1}$

$(A^{-1})^*A^{-1} = \overline{A^{-1}(A^*)^{-1}} = I^{-1}$

$(A^{-1})^*A^{-1} = \overline{A^{-1}(A^{-1})^*} = I$ (since $I^{-1} = I$)

Taking conjugate of the above equation we get

$\overline{(A^{-1})^*A^{-1}} = \overline{\overline{A^{-1}(A^{-1})^*}} = \bar{I}$

$\overline{(A^{-1})^*A^{-1}} = A^{-1}(A^{-1})^* = I$ (since $\bar{I} = I$)

$A^{-1}(A^{-1})^* = \overline{(A^{-1})^*A^{-1}} = I$

$\therefore A^{-1}$ is conjugate unitary matrix.

Theorem 2.5: Let A $\in$ C_{nxn}. If A is conjugate unitary matrix then (iA) is conjugate unitary matrix.

Proof: A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

Multiply by i of the above equation we get

$i(AA^*) = i(\overline{A^*A}) = iI$

$(iA)A^* = -\overline{(\imath A)^*A} = iI$

Again multiply by i,

-$(iA)(iA)^* = \overline{-(\imath A)^*(\imath A)} = i^2 I$

-$(iA)(iA)^* = \overline{-(\imath A)^*(\imath A)} = - I$

$(iA)(iA)^* = \overline{(\imath A)^*(\imath A)} = I$

$\therefore iA$ is conjugate unitary matrix.

Theorem 2.6: Let A $\in$ C_{nxn}. If A is conjugate unitary matrix then $\bar{A}$ is conjugateunitary matrix.

Proof : A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

Taking conjugate of the above equation we get

$\overline{AA^*} = \overline{\overline{A^*A}} = \bar{I}$

$\overline{AA^*} = \overline{\overline{A^*A}} = \bar{I}$

$\bar{A}\overline{A^*} = \overline{(\bar{A})^*\bar{A}} = I$ (since $\bar{I} = I$)

$\bar{A}(\bar{A})^* = \overline{(\bar{A})^*\bar{A}} = I$

$\therefore \bar{A}$ is conjugate unitary matrix.

Theorem 2.7: Let A $\in$ C_{nxn}. If A, B are conjugate unitary matrices and $AB^*+BA^* = \overline{A^*B+B^*A} = -I$ then (A+B) isconjugate unitary matrix.

Proof : If A isconjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$ and if B is conjugate unitary matrix $\Longrightarrow BB^* = \overline{B^*B} = I$

(i) We have to show that $(A+B)(A+B)^*=I$

$(A+B)(A+B)^* = (A+B)(A^*+B^*)$

$= AA^*+AB^*+BA^*+BB^*$

$= I+AB^*+BA^* + I$

$= AB^*+BA^* +2\ I$

$= - I + 2\ I$

$= I$ …..(2.1)

(ii) Next we have to show that $\overline{(A+B)^*(A+B)} = I$

$\overline{(A+B)^*(A+B)} = \overline{(A^*+B^*)(A+B)}$

$= \overline{A^*A + A^*B + B^*A + B^*B}$

$= \overline{I + A^*B + B^*A + I}$

$= \overline{2I + A^*B + B^*A}$

$= 2I\text{-}\overline{A^*B + B^*A}$

$= 2I\text{-}\ I= I$ …(2.2)

$\therefore$ From (2.1) and (2.2) we have $(A+B)(A+B)^*=\overline{(\mathrm{A+B})^*(\mathrm{A+B})}=I$

$\Rightarrow$ (A+B) is conjugate unitary matrix.

Theorem 2.8: Let A $\in C_{nxn}$. If A,B are conjugate unitary matrices then (AB) is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$ and if B is conjugate unitary matrix. $\Longrightarrow BB^* = \overline{B^*B} = I$

(i) We have to show $(AB)\ (AB)^* = I$

$(AB)\ (AB)^* = (AB)(B^*A^*)$

$= A\ (BB^*)A^*$

$= A\ IA^* = AA^* = I$ ……(2.3)

Next we have to show $\overline{(AB)^*(AB)} = I$

$\overline{(AB)^*(AB)} = \overline{(B^*A^*)(AB)}$

$= \overline{B^*(A^*A)B}$

$= \overline{B^*IB}$

$= \overline{B^*B} = I$ ……..(2.4)

$\therefore$ From (2.3) and (2.4) we have

$(AB)(AB)^*=\overline{(\mathrm{AB})^*(\mathrm{AB})} = I$

$\Rightarrow$ (AB) is conjugate unitary matrix.

Theorem 2.9: Let A $\in C_{nxn}$. If A is conjugate unitary matrix then (VA) is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

(i) We have to show $(VA)(VA)^* = I$

$(VA)(VA)^* = (VA)(A^*V^*)$

$= V\ (AA^*)V^*$

$= V\ IV($ since $V^* = V\ )$

$= V^2$ (since $V^2 = I$)

$= I$ …….(2.5)

Next we have to show $\overline{(VA)^*(VA)} = I$

$\overline{(VA)^*(VA)} = \overline{(A^*V^*)(VA)}$

$= \overline{A^*(V^*V)A}$

$= \overline{A^*(VV)A}$ (since $V^* = V$)

$= \overline{A^*V^2A}$

$= \overline{A^*IA}$ (since $v^2 = I$)

$= \overline{A^*A} = \bar{I} = I$ ..(2.6)

∴ From (2.5) and (2.6) we have

$(VA)(VA)^* = \overline{(VA)^*(VA)} = I$

⇒ (VA) is conjugate unitary matrix.

Theorem 2.10: Let A ∈ C_{nxn}. If A is conjugate unitary matrix then (AV) is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix ⇒ $AA^* = \overline{A^*A} = I$

(i) We have to show $(AV)(AV)^* = I$

$(AV)(AV)^* = (AV)(V^*A^*)$

$= (AV)(VA^*)$ (since $V^* = V$)

$= A(VV)A^*$

$= AV^2A^*$

$= AA^*$ (since $V^2 = I$)

$= I$ ….(2.7)

Next we have to show that $\overline{(AV)^*(AV)} = I$

$\overline{(AV)^*(AV)} = \overline{(V^*A^*)(AV)}$

$= \overline{V^*(A^*A)V} = \overline{V^*(I)V}$

$= \overline{V^*V} = \overline{VV}$ (since $V^* = V$)

$= \overline{V^2} = \bar{I} =$ (since $V^2 = I$)

$= I$ ..(2.8)

∴ From (2.7) and (2.8) we have

$(AV)(AV)^* = \overline{(AV)^*(AV)} = I$

⇒ (AV) is conjugate unitary matrix.

Theorem 2.11: Let A ∈ C_{nxn}. If A is conjugate unitary matrix then (AA^*) is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix ⇒ $AA^* = \overline{A^*A} = I$

(i) we have to prove $(AA^*)(AA^*)^* = I$

$(AA^*)(AA^*)^* = (AA^*)((A^*)^* A^*)$

$= (AA^*)(AA^*)$

$= I\,I = I^2$

$= I$ …..(2.9)

Next we prove $\overline{(AA^*)^*(AA^*)} = I$

$\overline{(AA^*)^*(AA^*)} = \overline{((A^*)^*A^*)(AA^*)}$

$= \overline{(AA^*)(AA^*)}$

$= \overline{I}\overline{I} = \overline{I}^2$

$= \bar{I} \quad = I$ …(2.10)

$\therefore$ From (2.9) and (2.10) we have $(AA^*)(AA^*) = \overline{(AA^*)^*(AA^*)} = I$

$\therefore$ (AA^*) is conjugate unitary matrix.

Theorem 2.12: Let $A \in C_{nxn}$. If A is conjugate unitary matrix then (A^*A) is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

(i) we have to prove $(A^*A)(A^*A)^* = I$

$(A^*A)(A^*A)^* = (A^*A)(A^*(A^*)^*)$

$= (A^*A)(A^*A)$

$= I\ I$

$= I^2 \quad = I$ ……..(2.11)

Next we prove that $\overline{(A^*A)^*(A^*A)} = I$

$\overline{(A^*A)^*(A^*A)} = \overline{(A^*(A^*)^*)(A^*A)}$

$= \overline{(A^*A)(A^*A)}$

$= \bar{I}\bar{I} = \bar{I}^2 = \bar{I}$

$= I$ ………(2.12)

$\therefore$ From (2.11) and (21.12) we have $(A^*A)(A^*A)^* = \overline{(A^*A)^*(A^*A)} = I$

$\Rightarrow$ (A^*A) isconjugate unitary matrix.

Theorem 2.13: Let $A \in C_{nxn}$. If A is conjugate unitary matrix then $(U^*\ A\ U)$ is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

(i) We have to show $(U^*\ A\ U)(U^*\ A\ U)^* = I$

$(U^*\ A\ U)(U^*\ A\ U)^* = (U^*\ A\ U)(U^*\ A^*(U^*)^*)$

$= (U^*\ A\ U)(U^*\ A^*\ U)$

$= (U^*\ A)(UU^*)(A^*\ U)$

$= (U^*\ A)(I)(A^*\ U)\ (\because UU^* = I)$

$= U^*(AA^*)U$

$= U^*(I)U$

$= U^*U = I$ ……….(2.13)

Next we have to prove $\overline{\left((U^*\mathrm{A\ U})^*(U^*\mathrm{A\ U})\right)} = I$

$\overline{\left((U^*\mathrm{A\ U})^*(U^*\mathrm{A\ U})\right)} = \overline{(U^*A^*(U^*)^*)(U^*AU)}$

$= \overline{(U^*A^*U)(U^*AU)}$

$= \overline{(U^*A^*)(UU^*)(AU)}$

$= \overline{(U^*A^*)(I)(AU)}(\because UU^* = I)$

$= \overline{(U^*A^*)(AU)}$

$= \overline{U^*(A^*A)U} \quad = \overline{U^*U} = \bar{I}$

$= I$ ….(2.14)

$\therefore$ From (2.13) and (2.14)

we have $(U^*\ A\ U)(U^*\ A\ U)^* = \overline{\left((U^*\mathrm{A\ U})^*(U^*\mathrm{A\ U})\right)} = I$

$\Rightarrow$ $(U^*\ A\ U)$ is conjugate unitary matrix.

Theorem 2.14: Let $A \in C_{nxn}$. If A is conjugate unitary matrix then (-iA) is conjugate unitary matrix.

Proof: If A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

$AA^* = \overline{A^*A} = I$

Multiply by –i on the above equation we have,

$(-i)AA^* = \overline{(-\imath)A^*A} = (-i)I$

$(-iA)A^* = \overline{-(-\imath A)^*A} = (-i)I$

Again multiply by $-i$ of the above equation we have,

$(-i)(-iA)A^* = \overline{(-\imath)(-(-\imath A)^*A} = (-i)(-i)I$

$-(-iA)(-iA)^* = \overline{-(-\imath A)^*(-\imath A)} = i^2 I$

$-(-iA)(-iA)^* = \overline{-(-\imath A)^*(-\imath A)} = -I$

$(-iA)(-iA)^* = \overline{(-\imath A)^*(-\imath A)} = I$

$\therefore$ $(-iA)$ is conjugate unitary matrix.

Conclusion

The conjugate unitary matrix was defined and theorems relating to characterizations of conjugate unitary matrices are derived. This concept may be applied to secondary conjugate unitary matrices, conjugate secondary unitary matrices etc.

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