ISSN 2231-346X (Print)
JUSPS-A Vol. 29(1), 46-51 (2017). Periodicity-Monthly
Section A
(Online) ISSN 2319-8044

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

Operations On Strict Fuzzy Graphs and Dominant Fuzzy Graphs

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http://dx.doi.org/10.22147/jusps-A/290108

Acceptance Date 26th Dece., 2016, Online Publication Date 2nd Janu., 2017

Abstract

In this paper, the authors study the various properties enjoyed by strict and dominant fuzzy graphs introduced in[3], by the authors. The usual operations on these fuzzy graphs are not extendable. For the first time such deviant nature is analysed.

Key words: Strict fuzzy graphs, dominant fuzzy graphs, vertex-edge matrix of fuzzy graph.

1. Introduction

In this paper, the authors continue their study about strict fuzzy graphs and dominant fuzzy graphs introduced in[3]. This paper is organized into four sections. Section one is introductory in nature. The basic concepts used are recalled in this section. In section two, many interesting properties enjoyed by these graphs are obtained. In section three, the various operations like composition, union and join of these fuzzy graphs are analysed and shown that they are not inherited in general. The final section gives the main research carried out in this paper. Here we recall the following definitions from [3].

Definition 1.1: The support of a fuzzy set μ of a set X is denoted by supp(μ) and is defined as $supp(\mu)=\{x \in X | \mu(x) > 0\}$.

Definition 1.2: A fuzzy graph (undirected and without loops) $G = (V, \mu, \rho)$ *is a nonempty finite set V together with a pair of functions* $\mu : V \to [0, 1]$ *and* $\rho : V \times V \to [0, 1]$ *such that* $\forall\ u, v \in V$, $\rho(u, v) \leq \min\{\mu(u), \mu(v)\}$, $\rho(u, v) = \rho(v,u)$ *and* $\rho(u, u) = 0$.

μ is called the fuzzy vertex set and ρ is called the fuzzy edge set of G.

Definition 1.3: A fuzzy graph $G = (V, \mu, \rho)$ *is said to be a complete fuzzy graph if* $\rho(u, v) = \min\{\mu(u), \mu(v)\}\ \forall\ u, v \in V$ *with* $u \neq v$.

Definition 1.4 :[2]. Let $G = (V, \mu, \rho)$ *be a fuzzy graph. Then the complement of G is denoted by* G^c *and is defined as* $G^c = (V, \mu, \rho^c)$ *where* $\rho^c(u, v) = \min\{\mu(u), \mu(v)\} - \rho(u, v)\ \forall\ u, v \in V$ *with* $u \neq v$ *and* $\rho^c(u, u) = 0\ \forall\ u \in V$.

Definition 1.5. Let $G = (V, \mu, \rho)$ *be a fuzzy graph with* $|V| = n$. *Let* $V = \{u_1, u_2, \ldots, u_n\}$ *be an ordered set of vertices. The* $v - e$ *matrix of the fuzzy graph G with respect to the ordered set V is denoted by* $m_V(G)$ *and is defined by* $m_V(G) = [a_{ij}]_{n \times n}$

where

$$a_{ij} = \begin{cases} \rho(u_i, u_j) & \text{if } i \neq j, \\ \mu(u_i) & \text{if } i = j. \end{cases}$$

Definition 1.6: A fuzzy graph G=(V, μ, ρ) is said to be a strict fuzzy graph if $\rho(u, v) < \min\{\mu(u), \mu(v)\}, \forall u, v \in V$.

Definition 1.7: Let G = (V, μ, ρ) be a fuzzy graph and let $V = \{u_1, u_2, \ldots, u_{|V|}\}$. *A fuzzy vertex* u_k *of G is said to be dominant (fuzzy) vertex of G if*

$$\mu(u_k) > \sum_{1 \le j \le |V|} \rho(u_k, u_j).$$

If every fuzzy vertex of G is dominant then G is said to be a dominant fuzzy graph.

For more properties and results about fuzzy graphs, strict fuzzy graphs and dominant fuzzy graphs, one can refer[1,3].

2. Main Results Related With Strict Fuzzy Graphs and Dominant Fuzzy Graphs :

In this section, we prove several interesting and important properties enjoyed by strict fuzzy graphs and dominant fuzzy graphs.

Proposition 2.1. A strict fuzzy graph G = (V, μ, ρ) with |V| = n > 1 is not complete.

Proof. For, if G is a strict fuzzy graph then by definition

$\rho(u, v) < \min\{\mu(u), \mu(v)\}, \forall u, v \in V$

and so $\rho(u, v) \neq \min\{\mu(u), \mu(v)\}, \forall u, v \in V$. Hence the result. □

Proposition 2.2. The diagonal entries of the v-e matrix of a strict fuzzy graph are strictly greater than all other entries in the respective rows and columns.

Proof. By the definition of a strict fuzzy graph, the strength of any fuzzy vertex is strictly greater than the strength of any fuzzy edge incident to it. The proof is evident from the fact that each diagonal entry denotes the strength of each fuzzy vertex and the rest of the entries in the corresponding row (or column) denote the strengths of the fuzzy edges incident to it.

Proposition 2.3. If G = (μ, ρ) is a dominant fuzzy graph, then G is also a strict fuzzy graph.

Proof. Let G = (μ, ρ) be a dominant fuzzy graph and let $V = \{u_1, u_2, \ldots, u_n\}$. Then $\forall\ 1 \le k \le n$, by definition,

$$\mu(u_k) > \sum_{1 \le j \le n} \rho(u_k, u_j).$$

To show: $\rho(u_i, u_j) < \min\{\mu(u_i), \mu(u_j)\} \forall u_i, u_j \in V$.

Assume that the contrary holds. Then there exist distinct fuzzy vertices u_1 and u_m such that $\rho(u_1, u_m) = \min\{\mu(u_i), \mu(u_m)\}$. Now since G is a dominant fuzzy graph, $\mu(u_1) > \sum_{1 \le j \le n} \rho(u_l, u_j) = \rho(u_l, u_m) + \sum_{\substack{1 \le j \le n \\ j \neq m}} \rho(u_l, u_j)$

This implies $\mu(u_l) > \rho(u_l, u_m)$. By similar argument, $\mu(u_m) > \rho(u_l, u_m)$. And so, $\rho(u_l, u_m) < \min\{\mu(u_l), \mu(u_m)\}$. This is a contradiction to the assumption. Therefore G is indeed a strict fuzzy graph. □

Proposition 2.4. A dominant fuzzy graph G = (V, μ, ρ) with |V| = n > 1 is not complete.

Proof. The proof follows from Proposition 2.1 and Proposition 2.3.

Proposition 2.5. A complete fuzzy graph G =(V, μ, ρ) with |V| = n > 1 is not a strict fuzzy graph and hence not dominant also.

Proof. The proof follows from Definition 1.3, Definition 1.6 and Proposition 2.3. □

Proposition 2.6. If G = (V, μ, ρ) is a strict fuzzy graph such that |V| > 1 and $supp(\rho) \neq V \times V \setminus \{(u, u) \mid u \in V\}$, *then* $G^c = (V, \mu, \rho^c)$, *the complement of G is not a strict fuzzy graph.*

Proof. Let G = (V, μ, ρ) be a strict fuzzy graph such that |V| > 1 and let $supp(\rho) \neq V \times V \setminus \{(u, u) \mid u \in V\}$.

This implies that there is atleast one fuzzy edge say (u, v) incident with two distinct fuzzy vertices u and v such that $\rho(u, v) = 0$.

Now, $\rho^c(u, v) = \min \{\mu(v), \mu(v)\} - \rho(u, v) = \min\{\mu(u), \mu(v)\}$. Therefore G^c is not a strict fuzzy graph.

Proposition 2.7. If $G = (V, \mu, \rho)$ is a strict fuzzy graph such that $|V| > 1$ and $supp(\rho) \neq V \times V \setminus \{(u, u) \mid u \in V\}$, then $G^C = (V, \mu, \rho^c)$ is also a strict fuzzy graph.

Proof. Let $G = (V, \mu, \rho)$ be a strict fuzzy graph with $|V| > 1$ and let $supp(\rho) = V \times V \setminus\{(u, u) \mid u \in V\}$. Let $u, v \in V$ and let $u \neq v$. Now, $\rho^c(u, v) = \min\{\mu(u), \mu(v)\} - \rho(u,v)$. By hypothesis, $\rho(u,v) > 0$; so $\rho^c(u, v) < \min \{\mu(u), \mu(v)\}$, $\forall\ u, v \in V$ with $u \neq v$. Therefore G^c is a strict fuzzy graph.

3. Operations on Strict Fuzzy Graphs and Dominant Fuzzy Graphs :

In this section, some operations on strict fuzzy graphs and dominant fuzzy graphs are analysed. It is seen in general the properties are not inherited however under certain conditions they happen to inherit their original properties.

Notation: Throughout this section, the edge between the two vertices u and v is denoted as uv rather than (u, v) because while taking the Cartesian product, a vertex of the product graph is, in fact an ordered pair.

Theorem 3.1. Let G_1 and G_2 be two fuzzy graphs such that each of their fuzzy vertices are dominant. Then it is not necessary that every fuzzy vertex of G, the composition of G_1 with G_2 is also dominant. □

Proof. The proof is by an example. Consider the fuzzy graphs G_1 and G_2 given in the following:

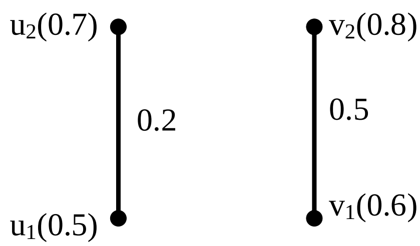

Figure 1: Dominant fuzzy graph G_1 and G_2

The v-e matrices of G_1 and G_2 are given by,

$$m_{V_1}(G_1) = \begin{array}{c} \\ u_1 \\ u_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \end{array} \qquad m_{V_2}(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.5 \\ 0.5 & 0.8 \end{bmatrix} \end{array}$$

$V_1 = \{u_1, u_2\}$ $V_2 = \{v_1, v_2\}$

$X_1 = \{u_1u_2\}$ $X_2 = \{v_1v_2\}$.

Now,

$V_1\times V_2 = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2)\}$,

$\mu_1 o\ \mu_2 = \mu_1 \times \mu_2$ on $V_1 \times V_2$

$(\mu_1 o\ \mu_2)(u_1, v_1) = (\mu_1\times \mu_2)(u_1, v_1) = \min\{\mu_1(u_1), \mu_2(v_1)\} = \min\{0.5, 0.6\} = 0.5$,

$(\mu_1 o\ \mu_2)(u_1, v_2) = (\mu_1\times \mu_2)(u_1, v_2) = \min\{\mu_1(u_1), \mu_2(v_2)\} = \min\{0.5, 0.8\} = 0.5$,

$(\mu_1 o\ \mu_2)(u_2, v_1) = (\mu_1\times \mu_2)(u_2, v_1) = \min\{\mu_1(u_2), \mu_2(v_1)\} = \min\{0.7, 0.6\} = 0.6$,

$(\mu_1 o\ \mu_2)(u_2, v_2) = (\mu_1\times \mu_2)(u_2, v_2) = \min\{\mu_1(u_2), \mu_2(v_2)\} = \min\{0.7, 0.8\} = 0.7$

$X = \{(u, u_2)(u, v_2) \mid u \in V_1, u_2v_2\in X_2\} \cup \{(u_1, w)(v_1, w) \mid w \in V_2, u_1v_1\in X_1\}$,

$= \{(u_1, v_1)(u_1, v_2), (u_1, v_1)(u_2, v_1), (u_1, v_2)(u_2, v_2), (u_2, v_1)(u_2, v_2)\}$.

For all $u \in V_1$, $\forall\ u_{-2}v_2 \in X_2$, $\rho_1\rho_2((u, u_2)(u, v_2)) = \min\{\mu_1(u), \rho_2(u_2, v_2)\}$,

$\forall\ w \in V_2\ \forall\ u_1v_1\in X_1$, $\rho_1\rho_2((u_1,w)(v_1, w)) = \min\{\mu_2(u), \rho_1(u_1, v_1)\}$.

$\rho_1\rho_2$ $((u_1, v_1)\ (u_1, v_2)) = \min\ \{\mu_1(u_1), \rho_2(v_1, v_2)\} = \min\{0.5, 0.5\} = 0.5$,
$\rho_1\rho_2$ $((u_1, v_1)\ (u_2, v_1)) = \min\ \{\mu_2(v_1), \rho_1(u_1, u_2)\} = \min\{0.6, 0.2\} = 0.2$,
$\rho_1\rho_2$ $((u_1, v_2)\ (u_2, v_2)) = \min\ \{\mu_2(v_2), \rho_1(u_1, u_2)\} = \min\{0.8, 0.2\} = 0.2$,
$\rho_1\rho_2$ $((u_2, v_1)\ (u_2, v_2)) = \min\ \{\mu_1(u_2), \rho_2(v_1, v_2)\} = \min\{0.7, 0.5\} = 0.5$.
$X^0 = \{(u, u_2)(u, v_2) \mid u \in V_1, u_2v_2 \in X_2\} \cup \{(u_1, w)(v_1, w) \mid w \in V_2, u_1v_1 \in X_1\}$
$\cup \{(u_1, u_2)\ (v_1, v_2) \mid u_1v_1 \in X_1, u_2 \neq v_2\}$,
$X^o \setminus X = \{(u_1, v_2)(u_2, v_1), (u_1, v_1)(u_2, v_2)\}$,
For all $(u_1, u_2)(v_1, v_2) \in X^o \setminus X$, $(\rho_1 o \rho_2)\ ((u_1, u_2)(v_1, v_2)) = \min\{\mu_2(u_2), \mu_2(v_2), \rho_1(u_1, v_1)\}$.
$(\rho_1 o \rho_2)\ ((u_1, v_2)(u_2, v_1)) = \min\{\rho_1(u_1,u_2), \mu_2(v_1), \mu_2(v_2)\} = \min\ \{0.2, 0.6, 0.8\} = 0.2$,
$(\rho_1 o \rho_2)\ ((u_1, v_1)(u_2, v_2)) = \min\{\rho_1(u_1,u_2), \mu_2(v_1), \mu_2(v_2)\} = \min\ \{0.2, 0.6, 0.8\} = 0.2$.

$$m_V(G) = \begin{array}{c} \\ (u_1,v_1) \\ (u_1,v_2) \\ (u_2,v_1) \\ (u_2,v_2) \end{array} \begin{array}{c} \begin{array}{cccc} (u_1,v_1) & (u_1,v_2) & (u_2,v_1) & (u_2,v_2) \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.6 & 0.5 \\ 0.2 & 0.2 & 0.5 & 0.7 \end{bmatrix} \end{array}$$

Each fuzzy vertex of G_1 and G_2 is dominant. However, it is not so in G. In fact none of the fuzzy vertices of G is dominant. □

Theorem 3.2. If $G_1 = (V_1, \mu_1, \rho_1)$ and $G_2 = (V_2, \mu_2, _2)$ are two dominant fuzzy graphs then G, the union of G_1 and G_2 is also a dominant fuzzy graph provided $V_1 \cap V_2 = \Phi$.

Proof. Let $G_1 = (V_1, \mu_1, \rho_1)$ and $G_2 = (V_2, \mu_2, \rho_2)$ be two dominant fuzzy graphs such that $V_1 \cap V_2 = \Phi$, Let V_1 be the ordered set of vertices $\{u_1, u_2, \ldots, u_m\}$ and let V_2 be the orderedset of vertices $\{v_1, v_2, ..., v_n\}$. Since $V_1 \cap V_2 = \Phi$, $V_1 \setminus V_2 = V_1$ and $V_2 \setminus V_1 = V_2$ and so,

$$(\mu_1 \cup \mu_2)(u) = \begin{cases} \mu_1(u) & if\ \ u \in V_1, \\ \mu_2(u) & if\ \ u \in V_2. \end{cases}$$

Again since $V_1 \cap V_2 = \Phi$, $X_1 \cap X_2 = \Phi$ and so $X_1 \setminus X_2 = X_1$ and $X_2 \setminus X_1 = X_2$. Now,

$$(\rho_1 \cup \rho_2)(uv) = \begin{cases} \rho_1(uv) & if\ \ uv \in X_1, \\ \rho_2(uv) & if\ \ uv \in X_2. \end{cases}$$

If G denotes the union of the two fuzzy graphs G_1 and G_2 and if V be the ordered set of vertices $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$, then the v-e matrix of G with respect to the ordered set of vertices V is given by,

$$m_V(G) = \begin{array}{c} \\ u_i \\ v_j \end{array} \begin{array}{c} \begin{array}{cc} u_i & v_j \end{array} \\ \begin{bmatrix} mV_1(G_1) & 0 \\ 0 & m_{V_2}(G_2) \end{bmatrix} \end{array}$$

Where $u_i \in V_1$, $\forall 1 \leq i \leq m$ and $v_j \in V_2$, $\forall\ 1 \leq j \leq n$.

Since the zero blocks do not contribute anything to the row-sum, the dominance of the vertices is not affected. Thus, if the fuzzy vertices of G_1 and G_2 are dominant vertices then so are the fuzzy vertices of G provided $V_1 \cap V_2 = \Phi$.

Example 3.1. Let G denote the union of G_1 and G_2 where G_1 and G_2 are as given in Theorem 3.1.

$$mV_1(G_1) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \end{array} \qquad mV_1(G_1) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \end{array}$$

Given,

$V_1 = \{u_1, u_2\}$, $V_2 = \{v_1, v_2\}$,
$X_1 = \{u_1u_2\}$ $X_2 = \{v_1v_2\}$.

And so,

$V = V_1 \cup V_2 = \{u_1, u_2, v_1, v_2\}$
$V_1 \cap V_2 = \Phi$,
$X = X_1 \cup X_2 = \{u_1u_2, v_1v_2\}$

$$(\mu_1 \cup \mu_2)(u) = \begin{cases} \mu_1(u) & \text{if } u \in V_1 \setminus V_2, \\ \mu_2(u) & \text{if } u \in V_2 \setminus V_1, \\ \max\{\mu_1(u), \mu_2(u)\} & \text{if } u \in V_1 \cap V_2. \end{cases}$$

$$(\rho_1 \cup \rho_2)\ (uv) = \begin{cases} \rho_1(uv) & \text{if } uv \in X_1 \setminus X_2, \\ \rho_2(uv) & \text{if } uv \in X_2 \setminus X_1, \\ \max\{\rho_1(uv), \rho_2(uv)\} & \text{if } uv \in X_1 \cap X_2. \end{cases}$$

$$m_V(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0 & 0 \\ 0.2 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.8 \end{bmatrix} \end{array}$$

It is easy to see that each fuzzy vertex of G is dominant.

Theorem 3.3. Let G_1 and G_2 be two fuzzy graphs such that all their fuzzy vertices are dominant. Then it is not necessary that every fuzzy vertex of G, the join of G_1 with G_2 is also dominant.

Proof. The proof is by an example. Let G denote the join of G_1 and G_2 where G_1 and G_2 are as given in Theorem 3.1.

$$mV_1(G_1) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \end{array} \qquad mV_2(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.5 \\ 0.5 & 0.8 \end{bmatrix} \end{array}$$

Given

$V_1 = \{u_1, u_2\}$ $V_2 = \{v_1, v_2\}$
$X_1 = \{u_1u_2\}$ $X_2 = \{v_1v_2\}$;

so,

$V_1 \cap V_2 = \Phi$,
$V = V_1 \cup V_2 = \{u_1, u_2, v_1, v_2\}$
$X_1 \cap X_2 = \Phi$,

X'= the set of all edges joining the vertices of V_1 and V_2 where we assume $V_1 \cap V_2 = \Phi$.
$X = X_1 \cup X_2 \cup X' = \{u_1u_2\} \cup \{v_1v_2\} \cup \{u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$
$= \{u_1u_2, v_1v_2, u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$.

$$(\mu_1 \cup \mu_2)\ (u) = \begin{cases} \mu_1(u) \ if \ u \in V_1, \\ \mu_2(u) \ if \ u \in V_2, \end{cases}$$

$$(\rho_1 \cup \rho_2)\ (uv) = \begin{cases} \rho_1(uv) & if \ uv \in X_1, \\ \rho_2(uv) & if \ uv \in X_2, \\ \min(\mu_1(u), \mu_2(v)\} & if \ uv \in X' \end{cases}$$

$$m_V(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.5 & 0.5 \\ 0.2 & 0.7 & 0.6 & 0.7 \\ 0.5 & 0.6 & 0.6 & 0.5 \\ 0.5 & 0.7 & 0.5 & 0.8 \end{bmatrix} \end{array}$$

Each fuzzy vertex of G_1 and G_2 is dominant. However, it is not so in G. In fact none of the fuzzy vertices of G is dominant.

4. Conclusion

In this paper, the authors have analysed the properties enjoyed by strict fuzzy graphs and dominant fuzzy graphs. It is shown that in general, the composition of two dominant fuzzy graphs need not be dominant. However the union of two dominant fuzzy graphs is dominant provided their underlying crisp vertex sets are mutually exclusive. But in the caseof join, this is not true in general.

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