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A Control Modelling Effect of Primary and Secondary Toxicants on theSurvival orExtinction of Resource based Population in Aquatic Environment: A Qualitative Approach

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Abstract

In this paper, a non-linear control mathematical model is proposed and analyzed to study the effect of primary and secondary toxicants on the survival or the extinction of resource based population in aquatic environment. The model is formulated by using system of non-linear ordinary differential equations. In the analysis, all the feasible equilibrium points of the system have been obtained. The conditions for local and non-linear stability of the non-trivial equilibrium points have been carried out using a suitable Lyapunov function. It may be observed from the stability conditions that the density of the resource biomass (zooplankton population) and resource based biological species (fish population) would co-exist if their growth rates were more than the death (removal) rates of both the species due to toxicants both primary and secondary and also when their carrying capacities are positive.

Key words : Resource biomass**,** Biological species, Toxicity, Emission, Aquatic environment.
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1. Introduction

Pollution material is introduced by man into the aquatic environment may be organic or inorganic wastes as byproducts of factories, sewage and sullage, household, garbage, petrochemical and oils, mining, dying industries, alcohol, acid production, detergents and radio-active pollution. The increasing amount of toxic elements in the aquatic environment caused by industrialization affects the structure and functions of eco-system.

It is well known that no population that if found in nature live in isolation therefore the study of the existence of two or more interacting species systems under the toxicant stress is important in order to conserve the biological population to maintain the stability of the ecosystem. For example, the industrial pollutants have affected the water, air and the land which in turn have affected the large number of the biological species. The domestic and transportation uses of the fossils fuels have created the air pollution, the discharges of the industries and households of the cities have damagingly polluted the water of rivers, lakes and coastal seas. The pollutants have affected the biological species directly as well as indirectly by deteriorating the resource biomass (zoo-plankton population) on which some biological species (fish) are dependent.

In recent years some investigations have been conducted to study the effects of toxicants (pollutants) on biological species[2-5]. Rai and Malviya[1] have discussed for the survival of resource based Industries when its resource is depleted by toxicants and augmented by precursor. They have suggest that environmental management system *i.e.* carbon emitter taxes apply on resources and its dependent industries so that equilibrium level is controlled to some extent from going down. It has been found that the carbon emitter taxes prove as disincentive to the emitters of the pollutants and its emission at source is checked and reduced and the resources and resource based industries can be maintained at a desired level.

Hallam and De Luna[8] proposed and analyzed a mathematical model to study the effects of toxicants on the biological population when the toxicants are emitted into the environment from external sources and they have also discussed at length the effect of a toxicant on population and showed that population will only persist if it exhibits a consistent potential for growth. Freedman and Shukla[7,9] in their model considered that the intrinsic growth rate of biological species decreases as the uptake concentration of the toxicant increases, while the carrying capacity of the species decreases as the concentration of the toxicant in the environment increases. Shukla and Dubey[6] proposed the model to study the simultaneous effects of two toxicants with different toxic concentrations on a biological population, in their model they assume that both toxicants are emitted from external sources. Shukla *et al.*[11] also modeled the effects of primary and secondary toxicants on renewable resources.

Mishra and Saxena[15] studied the effects of environmental pollution on the existence of interacting and dispersing biological population. They have shown in the resource based population system that the equilibrium state of resource bio mass (zoo-plankton population) remains stable at reduced level due to pollution but its magnitude further decreases due to the consumer's population (fish population). The equilibrium level of consumer's population (fish population) is reduced due to the decrease in equilibrium level of resource biomass (zoo-plankton population) on account of adverse effect of population. In the view of the above, we have proposed a dynamical model for conservation of resource-based population by controlling the emission rates of primary and secondary toxicant into the environment.

2. Mathematical Model

In this model, the formation of secondary toxicants has been considered in the growth equation of resource biomass (zoo-plankton population). It has been assumed that the primary toxicant reduces the carrying capacity of the environment of population and the secondary toxicant reduces the intrinsic growth rate of resource biomass (zoo-plankton population). In the growth equation of the predator population, it has been assumed that the secondary pollutant decreases the size of the consumer population (fish population) and the primary toxicant decreases the carrying capacity. The densities of resource biomass, biological species as well as the concentration of the primary and secondary pollutants in the aquatic environment are assumed to be governed by logistic equations. In view of the above assumptions, we propose the following model, governing the dynamics of theresource- based population, concentrations ofthe primary and secondary toxicant/pollutant.

$$\begin{aligned}
\frac{dN}{dt} &= r(B)N - \frac{r_0 N^2}{K(C)} - \alpha C_1 N \\
\frac{dB}{dt} &= s(N)B - \frac{s_0 B^2}{L(C)} - \alpha_1 C_1 B \qquad (2.1)\\
\frac{dC}{dt} &= Q - (\phi + a_{12})C \\
\frac{dC_1}{dt} &= a_{12}C - \phi_1 C_1
\end{aligned}$$

$$B(0) = B_0 \geq 0,\ C(0) = C_0 \geq 0,\ N(0) = N_0 \geq 0,\ C_1(0) \geq C_{10}.$$

taking:
$$r(B) = r_0 + r_{11}B$$
$$s(N) = s_0 - s_{12}N$$
$$K(C) = K_0 - K_2 C$$
$$L(C) = L_0 - L_{11}C$$

whereare $\phi,\ \phi_1,\ a_{12},\ \alpha,\ \alpha_1,\ r_0,\ s_0,\ r_{11},\ s_{12},\ K_0,\ K_2,\ L_0,\ L_{11}$ all positive constants.

Here N(t) is the density of the consumer's population (fish population), B(t) is the density of the resource biomass (Zoo plankton population), C(t) is the concentration of the primary toxicant introduced in the environment of this population and $C_1(t)$ is the concentration of the secondary toxicant, ϕ and ϕ_1 are natural washout rates, a_{12} is transformation rate.

In this model we have assumed the growth rate 'r' of the population decreases due to presence of toxicant concentration in N and carrying capacity of the environment is deteriorated by the presence of secondary toxicant in the environment. We also assumed that the primary toxicant may be externally introduced in to the environment at the rate Q which may be transformed to other harmful toxicant at the rate a_{12}. The primary toxicant in the environment is washed out with the rate ϕ. The population consumes the secondary toxicant from the environment indirect proportion to its concentration C_1. The toxicant concentration C_1 is washed out directly from the environment at the rate ϕ_1 and α_1 is the depletion rate of toxicant concentration C due to its conversion in to secondary toxicant α is the depletion rate coefficient of consumers population(fish population)due to concentration of primary and secondary toxicant.

In this model r(B) is the growth rate coefficient of the consumer's population and it increases as resource biomass density increases. The function K(C) is carrying capacity of aquatic habitat for consumer's population (fish population) which increases as the density of resource biomass (Zoo plankton) increases and decreases as the concentration of primary and secondary toxicant increases.

$$K(0) = K_0 > 0, \quad \frac{dK(C)}{dC} < 0, \quad \text{for} \quad C \geq 0. \qquad (2.1a)$$

where K_0 is the carrying capacity independent of C.

The function s(N) is growth rate of Zooplankton which decreases as N increases and hence

$$s(0) = s_0 > 0, \quad s'(N) < 0, \quad \text{for some } N \geq 0. \qquad (2.1b)$$

where s_0 is the growth rate coefficient independent of consumer's population (fish population).

The function $L(C)$ is the carrying capacity of resource bio mass (zooplankton population) which decreases as the concentration increases, we have

$$L(0) = L_0 > 0, \quad \frac{dL(C)}{dC} < 0, \quad \text{for} \quad C \geq 0. \qquad (2.1c)$$

where L_0 is the toxicant independent carrying capacity.

3. Equilibrium Analysis

The given model (2.1) has two non-negative real equillibria (Feasible equilibrium points) in N – B – C – C_1 space denoted by $E_0(0,0,C,C_1)$ and $E^*(N^*, B^*, C^*, C_1^*)$.

For, $E_0(0,0,C,C_1)$

$$C = \frac{Q}{\phi + a_{12}}, \; provided \; \phi + a_{12} \neq 0 \cdot$$

$$C_1 = \frac{a_{12}C}{\phi_1}.$$

The existence of E_0 is obvious.

Existence of Internal Equilibrium Point $E^*(N^*, B^*, C^*, C_1^*)$.

The interior equilibrium $E^*(N^*, B^*, C^*, C_1^*)$ is the solution of the following system of equations:

$$N^* = \frac{r(B)K(C) - \alpha C K(C)}{r_0} \qquad (3.2a)$$

$$B^* = \frac{s(N)L(C) - \alpha_1 C_1 L(C)}{s_0} \qquad (3.2b)$$

$$C^* = \frac{Q}{\phi + a_{12}}, \; provided \; \phi + a_{12} \neq 0 \qquad (3.2c)$$

$$C_1^* = \frac{a_{12}C}{\phi_1} \qquad (3.2d)$$

where

$$N^* > 0 \;\; if \;\; r(B) > \alpha C \;\; and \;\; K(C) > 0$$

$$B^* > 0 \;\; if \;\; s(N) > \alpha_1 C_1 \;\; and \;\; L(C) > 0$$

4. Stability Analysis :

Here we shall discuss the local stability of the equilibrium points. The local stability of thetrivial equilibrium point can be studied from variational matrices and for the non-trivial equilibrium point, suitable Lyapunov function is found.

4a. Local Stability Via Eigen Value Method :

To study the local stability behavior of equilibria, we compute the variational matrices corresponding to each equilibrium points. Let M_0 and M^* be the variational matrices corresponding to equilibrium points E_0 and E^* respectively.

$$M_0 = \begin{bmatrix} r_0 - \left(\frac{\alpha a_{12} Q}{\phi_1(\phi + a_{12})}\right) & 0 & 0 & 0 \\ 0 & s_0 - \left(\frac{\alpha_1 a_{12} Q}{\phi_1(\phi + a_{12})}\right) & 0 & 0 \\ 0 & 0 & -\phi - a_{12} & 0 \\ 0 & 0 & a_{12} & -\phi_1 \end{bmatrix},$$

and

$$M^* = \begin{bmatrix} -\frac{s_0 N^*}{K_0 - K_2 C^*} & -r_{11}N^* & -\frac{r_0 (N^*)^2 K_2}{(K_0 - K_2 C^*)^2} & -\alpha N^* \\ -s_{12}B^* & -\frac{s_0 B^*}{L_0 - L_{11}C^*} & -\frac{s_0 (B^*)^2 L_{11}}{(L_0 - L_{11}C^*)^2} & -\alpha_1 B^* \\ 0 & 0 & -\phi - a_{12} & 0 \\ 0 & 0 & a_{12} & -\phi_1 \end{bmatrix}.$$

From the matrix M_0, it is clear that $E_0(0,0,C,C_1)$ is a saddle point with stable manifold locally in the *C-C₁* space and unstable manifold locally in the *N-B* space if $r_0 - \alpha C_1 > 0$ and $s_0 - \alpha_1 C_1 > 0$.

The stability behavior of E^* is not obvious from M^*. However, in the following theorem we find sufficient condition for E^* to be locally asymptotically stable.

The following theorem gives the criteria for the local stability of E^* which can be proved by constructing a suitable Lyapunov function.

4b. Local Stability ViaLyapunov Method:

*Theorem 4.1:*If the following inequalities hold:

$$\frac{r_0 N^*}{\left[K(C^*)\right]} + \frac{r_0 N^* K_2}{2} - \frac{r_{11} N^*}{2} + \frac{\alpha N^*}{2} - \frac{s_{12} B^*}{2} > 0 \quad (4.1a)$$

$$\frac{s_0 B^*}{\left[L(C^*)\right]} - \frac{r_{11} N^*}{2} + \frac{\alpha_1 B^*}{2} - \frac{s_{12} B^*}{2} + \frac{s_0 \left(B^*\right)^2 L_{11}}{\left[L_1(C^*)\right]^2} > 0 \quad (4.1b)$$

$$\frac{r_0 N^* K_2}{\left[K(C^*)\right]} + \phi + \frac{a_{12}}{2} > 0 \quad (4.1c)$$

$$\frac{\alpha N^*}{2} + \phi_1 + \frac{\alpha_1 B^*}{2} + \frac{s_0 \left(B^*\right)^2 L_{11}}{\left[L_1(C^*)\right]^2} - \frac{a_{12}}{2} > 0 \quad (4.1d)$$

then $E^*(N^*, B, C^*, C_1^*)$ is locally asymptotically stable.

Proof of Theorem 4.1: First, we linearize the system (2.1) about $E^*(N^*, B, C^*, C_1^*)$ by using the following transformations

$N = N^* + n_1,\ B = B^* + n_2,\ C = C^* + n_3,\quad C_1 = C_1^* + n_4$

where n_1, n_2, n_3 and n_4 are small perturbations around E^* and on simplifying these equations, neglecting second order term of small perturbation, we get

$$\frac{dn_1}{dt} = r_{11} N^* n_2 - \frac{r_0 \left(N^*\right)^2 K_2 n_3}{\left[K(C^*)\right]^2} - \frac{r_0 N^* n_1}{K(C^*)} - \alpha N^* n_4 \quad (4.1e)$$

$$\frac{dn_2}{dt} = -s_{12} B^* n_1 - \frac{s_0 \left(B^*\right) n_2}{L(C^*)} - \frac{s_0 \left(B^*\right)^2 L_{11} n_3}{\left[L(C^*)\right]^2} - \alpha B^* n_4 \quad (4.1f)$$

$$\frac{dn_3}{dt} = -n_3 \phi - n_3 a_{12} \quad (4.1g)$$

$$\frac{dn_4}{dt} = a_{12} n_3 - \phi_1 n_4 \quad (4.1h)$$

Consider the positive definite function around E^*,

$$V = \frac{1}{2}\left(n_1^2 + n_2^2 + n_3^2 + n_4^2\right) \quad (4.1i)$$

We can show that the derivative of V with respect to t along the linearized system (4.1e) to (4.1h) is negative definite under the conditions of theorem (4.1).

Hence V is a Lyapunov function with respect to E^*, therefore E^* is locally asymptotically stable.

Conclusion

The conclusion drawn here suggests that emission of various kindsof toxicants in the environment must be controlled without further delay otherwise the survival of resource based biological species will be threatened.

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References

1. AlokMalviya and B. Rai, A control model in ecology for the survival of resources based industries when its resource is depleted by toxicant/pollutants and augmented precursors: A qualitative approach, *Journal of International Academy of Physical Sciences, 12,* 55-74 (2008).
2. Alok Malviya and Ankita Bajpai : A Control ecological model for the survival of plant biomass affected by an intermediate toxic product formed by uptake of a toxicant, *J. Jnanabha, 42,* 45-58 (2012).
3. B. Dubey and B. Das: Models for survival species dependent on resources in industrial environment, *J. Math. Anal. Appl., 231,* 374-396 (1999).
4. B. Dubey and J. B. Shukla: Simultaneous effects of two toxicants on biological species: A mathematical model, *J. Biol. Systems, 4,* 109-130 (1996).
5. B. Dubey and J.Hussain: Modelling the survival of species dependent on a Resource in a Polluted Environment: *J. Math. Biol., 7,* 187-210 (2006).
6. D. M. Thomas, T. W. Snell and S. M. Joffer: A control problem in a polluted environment, *Math. Biosci., 133,* 139-163 (1996).
7. J. B. Shukla and H. I. Freedman: Models for effect of toxicant in single species and predator- pray systems, *J. Math. Biol., 30,* 15-30 (1991).
8. J. T. De Luna and T. G. Hallam: Effects of toxicants

on populations: A qualitative approach IV. Resource-Consumer- Toxicant models, *Ecol. Modelling., 35,* 249-273 (1987).

9. J. B. Shukla, H. I. Freedman, V. N. Pal, O. P. Mishra, M. Agrawal and A. Shukla: Degradation and Subsequent Regeneration of a Forestry Resource: A Mathematical Model, *Ecol. Modelling, 44,* 219-229 (1999).
10. J. LaSalle and S. Lefschetz: *Stability by Lyapunov's Direct Method with applications*, Academic Press, New York, London (1961).
11. J.B. Shukla, A.K. Agarwal, P. Sinha and B. Dubey: Modeling effects of primary and secondary toxicants on renewable resources, Natural Resources Modeling, *16,* 99-120 (2003).
12. J.B. Shukla and B. Dubey: Modeling the depletion and conservation of forestry resources: Effect of population and pollution, *J. Math. Biol. 36,* 71-94 (1997).
13. M. Agarwal and Sapna Devi : A Resource-dependent competition Model: Effect of toxicants emitted from external sources as well as formed by precursors of competing species, *J. Math Biol., 12,* 751-766 (2011).
14. N.L. Miller : Stability analysis of toxic substance within aquatic ecosystem and their effect on aquatic population, *Ecological Modelling., 60,* 151-165 (1992).
15. O.P. Mishra and V.P. Saxena: Effect of environment pollution on the growth and existence of biological populations; Modelling and stability analysis, *Indian Journal of Pure and Applied Maths.* (1991).
16. P.D.N. Srinivasu: Control of environmental pollution to conserve a population, *Nonlinear Anal. Real World Appl. 3,* 397-411 (2002).
17. R.K. Trivedi : Aquatic pollution and Toxicology, *ABD Publishers* 70-79 (2001).