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On Double Integral Involving Generalized I-Function of Two Variables

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Abstract

The aim of this research paper is to evaluate a double integral involving generalized I-function of two variables.

Key words : I-Function, double integral

1. Introduction

The generalized I–function of two variables introduced by Goyal and Agrawal[1], will be defined and represented as follows:

$$I\begin{bmatrix}x\\y\end{bmatrix} = I^{m,n:m_1,n_1:m_2,n_2}_{p_i,q_i:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}[(a_j:\alpha_j,A_j)_{1,n}],[(a_{ji}:\alpha_{ji},A_{ji})_{n+1,p_i}]\\ [(b_j:\beta_j,B_j)_{1,n}],[(b_{ji}:\beta_{ji},B_{ji})_{1,q_i}]\end{matrix}\right.$$

$$\left.\begin{matrix}:[(c_j;\gamma_j)_{1,n_1}],[(c_{ji'};\gamma_{ji'})_{n_1+1,p_{i'}}];[(e_j;E_j)_{1,n_2}],[(e_{ji''};E_{ji''})_{n_2+1,p_{i''}}]\\ :[(d_j;\delta_j)_{1,m_1}],[(d_{ji'};\delta\gamma_{ji'})_{m_1+1,q_{i'}}];[(f_j;F_j)_{1,m_2}],[(f_{ji''};F_{ji''})_{m_2+1,q_{i''}}]\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,x^\xi y^\eta d\xi d\eta, \tag{1}$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi-B_j\eta)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\sum_{i=1}^{r}[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi-A_{ji}\eta)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi+B_{ji}\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\xi)}{\sum_{i'=1}^{r'}[\prod_{j=m_1+1}^{q_{i'}}\Gamma(1-d_{ji'}+\delta_{ji'}\xi)\prod_{j=n_1+1}^{p_{i'}}\Gamma(c_{ji'}-\gamma_{ji'}\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}\Gamma(1-e_j+E_j\eta)}{\sum_{i''=1}^{r''}[\prod_{j=m_2+1}^{q_{i''}}\Gamma(1-f_{ji''}+F_{ji''}\eta)\prod_{j=n_2+1}^{p_{i''}}\Gamma(e_{ji''}-E_{ji''}\eta)},$$

,

x and y are not equal to zero, and an empty product is interpreted as unity p_i, $p_{i'}$, $p_{i''}$, q_i, $q_{i'}$, $q_{i''}$, m, n, n_1, n_2, n_j and m_k are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''} \geq n_2 \geq 0$, $q_i \geq m > 0$, $q_{i'} \geq 0$, $q_{i''} \geq 0$, ($i = 1, \dots, r$; $i' = 1, \dots, r'$; $i'' = 1, \dots, r''$; $k = 1, 2$) also all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the ξ-plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ ($j = 1, \dots\dots\dots, m_1$) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ ($j = 1, \dots, n_1$), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ ($j = 1, \dots, n$) to the left of the contour.

The contour L_2 is in the η-plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j-F_j\eta)$ ($j=1,\dots\dots, n_2$) lie to the right, and the poles of $\Gamma(1 - e_j+E_j\eta)$ ($j = 1, \dots, m_2$), $\Gamma(1 - a_j+ \alpha_j\xi + A_j\eta)$ ($j = 1, \dots, n$) to the left of the contour. Also

$$R' = \sum_{j=1}^{p_i}\alpha_{ji} + \sum_{j=1}^{p_i'}\gamma_{ji'} - \sum_{j=1}^{q_i}\beta_{ji} - \sum_{j=1}^{q_{i'}}\delta_{ji'} < 0,$$

$$S' = \sum_{j=1}^{p_i}A_{ji} + \sum_{j=1}^{p_i''}E_{ji''} - \sum_{j=1}^{q_i}B_{ji} - \sum_{j=1}^{q_{i''}}F\delta_{ji'} < 0,$$

$$U = \sum_{j=n+1}^{p_i}\alpha_{ji} - \sum_{j=m+1}^{q_i}\beta_{ji} + \sum_{j=1}^{m_1}\delta_j - \sum_{j=m_1+1}^{q_{i'}}\delta_{ji'} + \sum_{j=1}^{n_1}\gamma_j - \sum_{j=n_1+1}^{p_i'}\gamma_{ji'} > 0, \tag{2}$$

$$V = -\sum_{j=n+1}^{p_i}A_{ji} - \sum_{j=m+1}^{q_i}B_{ji} - \sum_{j=1}^{m_2}F_j - \sum_{j=m_{21}+1}^{q_{i''}}F_{ji''} + \sum_{j=1}^{n_2}E_j - \sum_{j=n_2+1}^{p_i''}E_{ji''} > 0, \tag{3}$$

and $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$.

In our investigation we shall need the following results:

From Shrivastava [4, p.1, Eq. (1)]:

$$S_n^m[x] = \sum_{u=0}^{[n/m]}\frac{(-n)_{mu}}{u!}X^u A_{n,u}, \quad (n = 0, 1, 2, \dots), \tag{4}$$

where m is arbitrary positive integer, and the coefficients $A_{n,u}$ ($n, u \geq 0$) are arbitrary constants, real or complex.

2. Double Integral:

In this section, we shall establish following integral:

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}$$

$$\cdot\ {}_2F_1\left[{a,b; \atop 1+a-b;}\frac{(1+t)y}{1+ty}\right] J_s^{(\alpha,\beta)}(1-2x;k) S_f^e\left[cy^\gamma(1+ty)^\gamma(1-y)^{-2\gamma}\right]$$

$$\times I_{p_i,q_i:r\,:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}^{m,\ n\ :m_1,n_1\ :m_2,n_2}\left[\begin{matrix} z_1 x^{-\sigma}(1-x)^{\nu}\left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{-T} \\ z_2 \end{matrix}\right] dx\,dy$$

$$= \frac{2^a\{4(1+t)\}^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\,\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!}\sum_{u=0}^{[f/e]}\sum_{j=0}^{s}\frac{(-f)_{eu}}{u!}\frac{(\alpha+\beta+s+1)_{kj}(-s)_j c^u}{(\alpha+1)_{kj}\,j!\,\{4(1+t)\}^{\gamma u}}A_{f,u}$$

$$I_{p_i,q_i:r\,:p_{i'}+3,q_{i'}+4:r':p_{i''},q_{i''}:r''}^{m,\ n\ :m_1+2,n_1+3\ :m_2,n_2}\left[\begin{matrix} z_1\{4(1+t)\}^{T} \\ z_2 \end{matrix}\right|$$

$$\left.\begin{matrix} \dots\dots\dots,\dots\dots:(1-\mu,\nu),\left(\frac{1}{2}-\frac{a}{2}+\rho+\gamma u,T\right),\left(-\frac{a}{2}+b+\rho+\gamma u,T\right),\dots\dots:\dots\dots\dots,\dots\dots\dots \\ \dots\dots\dots,\dots\dots:(\lambda+kj,\sigma),(\rho+\gamma u,T),\dots\dots,(1-\lambda-\mu-kj,\nu-\sigma),(-a+b+\rho+\gamma u,T):\dots\dots\dots,\dots\dots\dots \end{matrix}\right], \tag{5}$$

The integral (5) is valid if the following sets of (sufficient) conditions are satisfied:

(i) e is arbitrary positive integer, and the coefficient $A_{f,u}$ ($f,u \geq 0$) are arbitrary constants, real or complex.

(ii) $|\arg z_1| < \frac{1}{2}U\pi$, $|\arg z_2| < \frac{1}{2}V\pi$, where U and V is given in (2) and (3) respectively.

(iii) $\mathrm{Re}(1+a-b) > 0$, $Re(\alpha) > -1$, $Re(\beta) > -1$, $t > -1$,

$\gamma \geq 0, \mathrm{Re}(\lambda+\sigma\xi)>0, Re(\rho+\nu\xi)>0,$
$\mathrm{Re}(\rho+\gamma u-T\xi)>0, \mathrm{Re}(1+a-2\rho-2\gamma u+2T\xi)>0.$

(iv) Re(1 — 2b) > 0.

Proof:

To establish (5), we use the series representation of $S_f^e[x]$ as given by (4) and for the generalized I–function of two variables we use Mellin-Barnes types of contour integral as given in (1), on the left-hand side of (5), change the order of integration and summation (which is justified under the conditions given with (5)), we then obtain

$$\text{Left-hand side of (5)} = \sum_{u=0}^{[f/e]} \frac{(-f)_{e_u}}{u!} c^u A_{f,u}$$
$$\cdot \left[\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\,\theta_2(\xi)\theta_3(\eta) z_1{}^{\xi} z_2{}^{\eta}\right.$$
$$\cdot \int_0^1 x^{\lambda-\sigma\xi-1}\,(1-x)^{\mu+\nu\xi-1} J_s^{(\alpha,\beta)}(1-2x;k)dx$$
$$\cdot \int_0^1 y^{\rho+\gamma u-T\xi-1}\,(1-y)^{a+2T\xi-2\gamma u-2\rho}(1+ty)^{\rho-a+\gamma u-T-1}$$
$$\cdot\ {}_2F_1\left[{a,b; \atop 1+a+b;}\frac{(1+t)y}{1+ty}\right]dy\Big]d\xi d\eta.$$

Now using known results [2, p.118, Eq.(3.3.1)] and [3, p.254, Eq.(2.1)], and interpreting the resulting contour integral as the generalized I–function of two variables, we once get the right-hand side of (5).

3. Particular Cases:

I. On specializing the parameters in main integral, we get following integral in terms of I-function of one variable:

$$\int_0^1\int_0^1 x^{\lambda-1}\,(1-x)^{\mu-1} y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}$$
$$\cdot\ {}_2F_1\left[{a,b; \atop 1+a-b;}\frac{(1+t)y}{1+ty}\right] J_s^{(\alpha,\beta)}(1-2x;k) S_n^m[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}]$$
$$\cdot\ I_{P_i,Q_i:r}^{M,N}[zx^{-\sigma}(1-x)^{\nu}\{\frac{y(1+ty)}{(1-y)^2}\}^{-T}]dx\,dy$$
$$= \frac{2^a\{4(1+t)\}^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\,\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!}\sum_{u=0}^{[n/m]}\sum_{j=0}^{s}\frac{(-n)_{m_u}}{u!}$$
$$\cdot \frac{(\alpha+\beta+s+1)_{kj}(-s)_j c^u}{(\alpha+1)_{kj}\, j!\,\{4(1+t)\}^{\gamma u}} A_{n,u} I_{P_i+3,Q_i+4:r}^{M+2,N+3}[z\{4(1+t)\}^{T}|{A \atop B}], \tag{6}$$

where

$A = (1-\mu,\nu),\left(\frac{1}{2}-\frac{a}{2}+\rho+\gamma u,T\right),\left(-\frac{a}{2}+b+\rho+\gamma u,T\right),\ldots..$

$B = (\lambda+kj,\sigma),(\rho+\gamma u,T),\ldots\ldots,(1-\lambda-\mu-kj,\nu-\sigma),(-a+b+\rho+\gamma u,T).$

The integral (6) is valid if the following sets of (sufficient) conditions are satisfied:

(i) m is arbitrary positive integer, and the coefficient $A_{n,u}$ (n,u≥0) are arbitrary constants, real or complex.

(ii) $\Omega = \sum_{j=1}^{N}\alpha_j - \sum_{j=n+1}^{P_i}\alpha_{ji} + \sum_{j=1}^{M}\beta_j - \sum_{j=n+1}^{Q_i}\beta_{ji} > 0,$ and |arg z| < (1/2), Ωπ, ∀ i ∈ (1, 2, …, r).

(iii) $\mathrm{Re}(1+a-b)>0, Re(\alpha)>-1, Re(\beta)>-1, t>-1,$

$\gamma \geq 0, \mathrm{Re}(\lambda + \sigma\xi) > 0, Re(\rho + \nu\xi) > 0,$
$\mathrm{Re}(\rho + \gamma u - T\xi) > 0, \mathrm{Re}(1 + a - 2\rho - 2\gamma u + 2T\xi) > 0.$

(iv) Re (12b) > 0.

II. On choosing r = 1 in the integral (6), we get following integral in terms of H-function of one variable:

$$\int_0^1 \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} y^{\rho-1} (1-y)^{a-2\rho} (1+ty)^{\rho-a-1}$$

$$\cdot\ {}_2F_1\left[{}^{a,b;}_{1+a-b;} \frac{(1+t)y}{1+ty}\right] J_s^{(\alpha,\beta)}(1-2x;k) S_n^m[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}]$$

$$.\ H_{P,Q}^{M,N}\left[zx^{\sigma}(1-x)^{-\nu}\left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{-T}\Big|{}^{(a_j,\alpha_j)_{1,P}}_{(b_j,\beta_j)_{1,Q}}\right]dx\,dy$$

$$= \frac{2^a\{4(1+t)\}^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\,\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right) s!} \sum_{u=0}^{[n/m]} \sum_{j=0}^{s} \frac{(-n)_{mu}}{u!}$$

$$.\ \frac{(\alpha+\beta+s+1)_{kj}(-s)_j c^u}{(\alpha+1)_{kj}\, j!\,\{4(1+t)\}^{\gamma u}} A_{n,u} H_{P+3,Q+4}^{M+2,N+3}\left[z\{4(1+t)\}^{T}\Big|{}^{A}_{B}\right], \tag{7}$$

where

$A = (1-\mu,\nu), \left(\frac{1}{2} - \frac{a}{2} + \rho + \gamma u, T\right), \left(-\frac{a}{2} + b + \rho + \gamma u, T\right), \ldots..$

$B = (\lambda + kj, \sigma), (\rho + \gamma u, T), \ldots\ldots, (1 - \lambda - \mu - kj, \nu - \sigma), (-a + b + \rho + \gamma u, T).$

The integral (7) is valid if the following sets of (sufficient) conditions are satisfied:

(i) m is arbitrary positive integer, and the coefficient $A_{n,u}$ (n,u≥0) are arbitrary constants, real or complex.

(ii) $\Omega = \sum_{j=1}^{N} \alpha_j - \sum_{j=n+1}^{P} \alpha_j + \sum_{j=1}^{M} \beta_j - \sum_{j=n+1}^{Q} \beta_j > 0,$ and | arg z | < (1/2) Ωπ.

(iii) $\mathrm{Re}(1 + a - b) > 0, Re(\alpha) > -1, Re(\beta) > -1, t > -1,$
$\gamma \geq 0, \mathrm{Re}(\lambda + \sigma\xi) > 0, Re(\rho + \nu\xi) > 0,$
$\mathrm{Re}(\rho + \gamma u - T\xi) > 0, \mathrm{Re}(1 + a - 2\rho - 2\gamma u + 2T\xi) > 0.$

(iv) Re(1–2b) > 0.

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