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Characterizaton and Theorems on Quaternion Doubly Stochastic Matrices

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Abstract

The concepts of quaternion symmetric doubly stochastic are developed, basic theorems and some results for these matrices and characterization are analyzed with examples.

Key words : Doubly stochastic matrix, quaternion symmetric doubly stochastic matrix, quaternion orthogonal symmetric doubly stochastic matrix, centro doubly stochastic matrix.

Subject code classification:15B99, 15A51

Introduction

The concepts of quaternion symmetric doubly stochastic matrix are applied[1-4]. In this paper, the quaternion symmetric doubly stochastic matrix is developed in quaternion matrices. Denoted by A^T is the transpose of A and A^* is the conjugate transpose of A.

Definition (1)[5]

Suppose $A = (a_{ij})_{n\times n}$ is a doubly stochastic matrix such that, A matrix $A = (a_{ij})_{n\times n}$ is called a doubly stochastic matrix if $\sum_{i=1}^{n} a_{ij} = 1$ and $\sum_{j=1}^{n} a_{ij} = 1$ and all $a_{ij} \geq 0$

1. QUATERNION SYMMETRIC DOUBLY STOCHASTIC MATRIX.

Definition 1.1

A matrix $A \in H^{n\times n}$ is said to be quaternion symmetric doubly stochastic if $A^T = A$ and $\sum_{i=1}^{n} a_{ij} = 1,\ j = 1, 2, ...n$ and $\sum_{j=1}^{n} a_{ij} = 1,\ i = 1, 2, ...n$ and all $a_{ij} \geq 0$ (or) if A is doubly stochastic and also symmetric then it is called a quaternion symmetric doubly stochastic matrix.

Theorem 1.1

Let A be a square matrix in $H^{n\times n}$. Then A is quaternion symmetric doubly stochastic iff $A = A^T$.

Proof:

Let $A = (a_{ij})$ be an n×n matrix. Then $A^T=(b_{ij})$ is an n×n matrix.Where $b_{ij} = a_{ji}$ for all i, j.

$\Longrightarrow$ Let A is quaternion symmetric doubly stochastic. Then $a_{ij} = a_{ji}$ for all i, j from the definition, $a_{ij} = b_{ij}$ for all i, j.

Therefore,

$$A = A^T.$$

Let $A = A^T$ (Then $a_{ij} = b_{ij}$ for all i, j
$= a_{ij}$ for all i, j.)

$\Longrightarrow$ A is quaternion symmetric doubly stochastic matrix

$\Longrightarrow$ $A^T = A$ then $b_{ij} = a_{ij}$ for all i, j
$= b_{ji}$ for all i, j

$\Longrightarrow$ A^T is quaternion symmetric doubly stochastic matrix.

Theorem 1.2:

If A and B are n×n quaternion symmetric doubly stochastic matrices, then

(1) $\frac{1}{2}(A+B)^T = \frac{1}{2}(A^T+B^T)$

(2) $(kA)^T = kA^T$, where k is scalar are also quaternion symmetric matrices.

Proof:

(1) Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be quaternion symmetric doubly stochastic matrices.

Then ½ (A+B) is an n×n quaternion symmetric doubly stochasticmatrix.

Since A^T and B^T are n×n quaternion symmetric doubly stochastic matrices then $\frac{1}{2}(A^T+B^T)$ is also n×n quaternion symmetric matrix. Thus $\frac{1}{2}(A+B)^T$ and $\frac{1}{2}(A^T+B^T)$ are of same type

$(i,j)^{th}$ entry of $\frac{1}{2}(A+B)^T$ = $(j,i)^{th}$ entry of $\frac{1}{2}(A+B) = \frac{1}{2}(a_{ij} + b_{ji})$

= ½ {$(j,i)^{th}$ entry of A + $(j,i)^{th}$ entry of B}

= {$(i,j)^{th}$ entry of A^T + $(i,j)^{th}$ entry of B}

= ½ {$(i,j)^{th}$ entry of (A^T+B^T)}

$\Longrightarrow$ $\frac{1}{2}(A+B)^T = \frac{1}{2}(A^T+B^T)$

(2) Let $A = (a_{ij})_{n\times n}$quaternion symmetric doubly stochastic matrix then $(kA)_{n\times n}$ quaternion symmetric doubly stochastic matrix. wherek is Scalar and hence also $(kA)^T_{n\times n}$ quaternion symmetric matrix. Since $(A^T)_{n\times n}$ quaternion symmetric doubly stochastic matrix and also (kA)) quaternion symmetric matrix. Hence $(kA)^T$ and (kA^T) are of the same type.

Also $(i,j)^{th}$ entry of $(kA)^T$=(j,i) entry of (kA)

= k a_{ij}

= k $(j,i)^{th}$ entry of A.

= k $(i,j)^{th}$ entry of A^T

= $(i,j)^{th}$ entry of kA^T.

$(kA)^T = kA^T$, where k is scalar.

Example1.1: To prove $\frac{1}{2}(A+B)^T = \frac{1}{2}(A^T+B^T)$.

Let

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$B = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

½ $(A+B)^T$= ½ (A^T+B^T) = ½ (A + B)

$$\tfrac{1}{2}(A+B)^T = \tfrac{1}{2}\begin{pmatrix} 2+3i+3j & 4-2i-2j & -4-i-j \\ 4-2i-2j & -9+j & 7+2i+j \\ -4-i-j & 7+2i+j & -1-i \end{pmatrix}$$

$$A+B = \begin{pmatrix} 2+3i+3j & 4-2i-2j & -4-i-j \\ 4-2i-2j & -9+j & 7+2i+j \\ -4-i-j & 7+2i+j & -1-i \end{pmatrix}$$

½ $(A+B)^T$ is an quaternion symmetric doubly stochastic matrices.
Hence proved ½ $(A + B)^T$ = ½ (A^T+B^T) = ½ (A + B)

Property 1.1:

If $A \in H^{n\times n}$ is quaternion symmetric doubly stochastic matrix then A^n is also quaternion symmetric doubly stochastic matrix for positive integer n

Proof:

Let $A = (a_{ij})$ be an n×n matrix. Then $A^n = (b_{ji})$ is an n×n matrix, where $b_{ij} = a_{ij}$. for all i, j.

$\Rightarrow$ **:** Suppose A is quaternion symmetric doubly stochastic.

Then $a_{ij} = a_{ij}$ for all i, j from definition

$= b_{ij}$ for all i, j.

Therefore $A = A^T$

$\Leftarrow$ **:** Suppose $A = A^T$

Then $a_{ij} = b_{ij}$ for all i, j.

$= a_{ji}$ for all i, j.

$\Rightarrow$ A is quaternion symmetric doubly stochastic matrix.

Example 1.2: quaternion symmetric doubly stochastic matrices.

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$A = A^T$

$\Rightarrow$ A is quaternion symmetric doubly stochastic matrix.

Property 1.2:

Products of any two quaternion symmetric doubly stochastic matrices are not an quaternion symmetric doubly stochastic matrix if and only if quaternion symmetric doubly stochastic matrix is non – commutative.

Proof:

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$B = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3+k & -4-i-k & 2 \\ -16+6i+7j & 30+7j & -12-5i \\ 14+2i & -20 & 11-i+j \end{pmatrix}$$

Theorem 1.3:

If A and B are n×nq uaternion symmetric doubly stochastic matrices then $(AB)^T \neq B^T A^T$ is not a quaternion symmetric doubly stochastic matrix.

Proof:

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ quaternion symmetric doubly matrices then $AB = (c_{ji})$ is an n×n quaternion symmetric doubly stochastic matrix where, $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Let$(AB)^T = (d_{ij})$ where $d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$ the $(AB)^T_{n\times n}$ quaternion symmetric doubly stochastic matrix.

Let $A^T = (e_{ij})$ where $e_{ij} = a_{ji}$ and $B^T = (f_{ij})$ where $f_{ij} = b_{ji}$. Since $(B)^T_{n\times n}$ and $(A)^T_{n\times n}$ quaternion symmetric doubly stochastic matrices respectively. Hence $(B^TA^T)_{n\times n}$ quaternion symmetric doubly stochastic matrix. Thus $(AB)^T$ and B^TA^T are of same type.

quaternion does not satisfy commute properly.

Let $B^TA^T = (g_{ij})$ where $g_{ij} = \sum_{k=1}^{n} f_{ik} e_{kj}$

Also $(i,j)^{th}$ entry of $(AB)^T = d_{ij} = \sum_{t=1}^{n} A_t = \sum_{k=1}^{n} e_{kj} f_{ik}$ $[e_{ij} = a_{ij}$ & $f_{ij} = b_{ij}]$

$= \sum_{k=1}^{n} f_{ik} e_{kj} = g_{ij} =$ $(i,j)^{th}$ entry of B^TA^T. Thus $(AB)^T \neq B^TA^T$.

Theorem 1.4:

If A and B are n×n quaternion symmetric doubly stochastic matrices, then

(1) ½(A+B) is quaternion symmetric doubly stochastic matrix.

(2) (kA) is quaternion symmetric matrix, where K is scalar.

(3) ½ (AB +BA) is quaternion symmetric doubly stochastic matrix.

(4) AB is quaternion symmetric doubly stochastic matrix if and only AB ≠ BA.

Proof:

Since A and B are quaternion symmetric doubly stochastic matrices, so $A=A^T$ and $B=B^T$

(1) $\frac{1}{2}(A+B)^T = \frac{1}{2}(A^T+B^T) = \frac{1}{2}(A+B)$

$\Rightarrow \frac{1}{2}(A+B)$ is quaternion symmetric doubly stochastic matrix [see Ex:1.1]

(2) $(kA)^T = KA^T = KA$, where K is scalar.

$\Rightarrow$ (kA) is quaternion symmetric matrix, where K is scalar.

(3) $\frac{1}{2}(AB+BA)^T = \frac{1}{2}((AB)^T+(BA)^T) =$

$\frac{1}{2}(B^TA^T+A^TB^T) = \frac{1}{2}(BA+AB)$

$= \frac{1}{2}(AB+BA)$.

$\Rightarrow \frac{1}{2}(AB+BA)$ is quaternion symmetric doubly stochastic matrix.[6]

(4) Suppose AB is not quaternion symmetric doubly stochastic matrix, then $(AB)^T = AB$

(i.e,) $(AB)^T=AB \Rightarrow B^TA^T = AB \Rightarrow BA \neq AB$

$AB \neq BA$.

$\Leftarrow$ Suppose $AB \neq BA$, then $(AB)^T \neq (BA)^T = A^TB^T \neq AB$ is not an quaternion Symmetric doubly stochastic matrix.

Property 1.3: If $A,B \in H^{n\times n}$ then $(AB)^n \neq A^nB^n$ for $n > 1$

Example 1.3:

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$B = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3+k & -4-i-k & 2 \\ -16+6i+7j & 30+7j & -12-5i \\ 14+2i & -20 & 11-i+j \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -2 & 6 & 5 \\ 6 & 36 & 23 \\ 5 & 23 & 5 \end{pmatrix} \quad B^2 = \begin{pmatrix} -4 & 6 & 5 \\ 6 & 8 & 3 \\ 5 & 3 & 1 \end{pmatrix}$$

$$AB^2 = \begin{pmatrix} 8 & 18 & 4 \\ 171 & 851 & 169 \\ 194 & 400 & 121 \end{pmatrix}$$

$$A^2B^2 = \begin{pmatrix} 3 & 51 & 13 \\ 307 & 293 & 161 \\ 143 & 229 & 99 \end{pmatrix}$$

$(AB)^2 \neq A^2B^2$.

In general $(AB)^n \neq A^nB^n$.

Let as assume that this is true for n-1

$$(\mathrm{AB})^{n-1} \neq A^{n-1}B^{n-1}$$
$$(\mathrm{AB})^{n} = (AB)^{n-1}\,(AB)$$
$$\neq (AB)^{n-1}\,(BA)$$
$$\neq B^{n-1}\,(A^{n-1}B)A$$
$$\neq B^{n-1}BA^{n-1}A$$
$$\neq B^{n}A^{n}$$

In general $(\mathrm{AB})^{n-1} \neq A^{n-1}B^{n-1}$ its true for n > 1

quaternion symmetric matrices does not satisfy commutative property.

Property 1.4:

If A,B $\in H^{n\times n}$ are quaternion symmetric doubly stochastic matrices then A+B = 2c where C is another quaternion. Symmetric doubly stochastic matrix (or) The sum of symmetric doubly stochastic matrices of same order is twice the quaternion symmetric doubly stochastic matrix. (or) If A_1, A_2, ……$A_n \in H^{n\times n}$, then $\sum_{t=1}^{n} A_t$ is quaternion symmetric doubly stochastic matrices multiplied by 'n'

Proof:

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$B = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 2+3i+3j & 4-2i-2j & -4-i-j \\ 4-2i-2j & -9+j & 7+2i+j \\ -4-i-j & 7+2i+j & -2-i \end{pmatrix}$$

$$A+B = \begin{pmatrix} 1+\frac{3}{2}i+\frac{3}{2}j & 2-i-j & -2-i/2-j/2 \\ 2-i-j & -9/2+j/2 & 7/2+i+j/2 \\ -2-i/2-j/2 & 7/2+i+j/2 & -1-i/2 \end{pmatrix}$$

A+B = 2c

$$C = \begin{pmatrix} 1+\frac{3}{2}i+\frac{3}{2}j & 2-i-j & -2-i/2-j/2 \\ 2-i-j & -9/2+j/2 & \frac{7}{2}+i+j/2 \\ -2-i/2-j/2 & \frac{7}{2}+i+j/2 & -1-i/2 \end{pmatrix}$$

Theorem 1.5: If A is aquaternion symmetric doubly stochastic matrix, then ½(A+A^T) is quaternion symmetric doubly stochastic matrix[6].

Poof:

$\frac{1}{2}[(A+A^T)]^T = \frac{1}{2}[A^T+(A^T)]^T = \frac{1}{2}[A^T+A] = [(A^T)^T = A]$

$\Rightarrow \frac{1}{2}\,[A^T+A]$

Where $[(A^T)^T = A]$ is quaternion symmetric doubly stochastic matrix.

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$(A^T)^T = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$(A^T)^T = A$

To prove $\frac{1}{2}\,(A + A^T)$ is quaternion (or) symmetric doubly stochastic matrix.

$\frac{1}{2}\,(A + A^T)^T$

$$(A+A^T) = \begin{pmatrix} 2+2i+4j & 4-2i-2j & -4-2j \\ 4-2i-2j & -12 & 10+2i+2j \\ -4-2j & 10+2i+2j & -4-2i \end{pmatrix}$$

$$\frac{1}{2}(A + A^T)^T = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$(A + A^T)^T = \begin{pmatrix} 2+2i+4j & 4-2i-2j & -4-2j \\ 4-2i-2j & -12 & 10+2i+2j \\ -4-2j & 10+2i+2j & -4-2i \end{pmatrix}$$

$\frac{1}{2}(A + (A^T))^T$ is also an quaternion symmetric doubly stochastic matrix.

Property 1.5:

If $A \in H^{n\times n}$ is quaternion symmetric doubly stochastic matrix, then $\frac{1}{2}\,(A + A^T) = A$

Proof:

$\frac{1}{2}\,(A+A^T) = (2A)/2$ (or) $(2A^T)/2$

Where $A^T = A$

$= A$ or A^T

Example 1.4:

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$A+A^T = \begin{pmatrix} 2+2i+4j & 2+i-2j & -4-2j \\ 2+i-2j & -12 & 10+2i+2j \\ -4-2j & 10+2i+2j & -4-2i \end{pmatrix}$$

$$= 2\begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$\frac{1}{2}(A + A^T) = 2(A)/2$

$\frac{1}{2}(A + A^T) = A$. Hence proved.

Property 1.6: If $A \in H^{n\times n}$ is quaternion symmetric doubly stochastic matrix then $(A - A^T)$ is null matrix.

Proof:

If A is quaternion symmetric doubly stochastic matrix then $A^T = A$.

Hence $(A - A^T) = 0$ if $A^T = A$.

$$A = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1+i+2j & 2-i-j & -2-j \\ 2-i-j & -6 & 5+i+j \\ -2-j & 5+i+j & -2-i \end{pmatrix}$$

$$A - A^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(A - A^T)$ is an null matrix. Hence proved.

Definition 1.2:

A square matrix A is said to be as quaternion orthogonal symmetric doubly stochastic matrix if $AA^T = A^TA = I$

Theorem:1.6

If A is quaternion orthogonal symmetric doubly stochastic matrix, then A^T is also quaternion orthogonal symmetric doubly stochastic matrix.

Proof:

since A is quaternion orthogonal symmetric doubly stochastic matrix,

$AA^T = A^TA = I$. therefore, $(A^T)^TA^T = A^T(A^T)^T$

$AA^T = A^TA = I$

$\Rightarrow A^T$ is quaternion orthogonal symmetric doubly stochastic matrix.

Example1.5:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Property 1.7

In particular case of

$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc are doubly stochastic matrices then they are all quaternion orthogonal symmetric doubly stochastic matrices (i.e,) $A_2^{\,2} = I_2$ and $A_3^{\,2} = I_3$

Definition 1.3:

For any B $\in H^{n\times n}$ all doubly stochastic matrix is said to centro doubly stochastic matrix (or) centro bi stochastic matrix if B = Jn B Jn, where Jn is a exchange matrix.

Example 1.6 :

$$B = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$J_3\, B\, J_3 = \begin{pmatrix} 1+2i+j & 2-i-j & -2-i \\ 2-i-j & -3+j & 2+i \\ -2-i & 2+i & 1 \end{pmatrix}$$

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