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$q-k$ - Hermitian doubly Stochastic, $q-s$ - Hermitian Doubly Stochastic and $q-s-k$ Hermitian Doubly Stochastic Matrices

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Abstract

The basic concepts and theorems of $q-k$ - hermitian doubly stochastic, $q-s$ hermitian doubly stochastic and $q-s-k$ - hermitian doubly stochastic matrices are introduced with examples.

Key words : $q-k$ - hermitian Doubly Stochastic, $q-s$ - hermitian Doubly Stochastic and $q-s-k$ - hermitian doubly stochastic matrices.

Subject code classification:15B99, 15A51.

Introduction

We have already seen the concept of hermitian doubly stochastic matrices. In this paper, introduce of the hermitian doubly stochastic matrix is developed in quaternion matrices. Recently Hill and Waters[2] have developed a theory of k-real and k-hermitian matrices as a generalization of s-real and s-hermitian matrices. Ann Lee[1] has initiated The study of secondary hermitian matrices, that is matrices whose entries are symmetric about the secondary diagonal. Ann Lee[1] has show that the matrix A, the usual conjugate A and secondary conjugate A and secondary conjugate of A are related as $\overline{A}=VA^{*}V$ and $A^{*}=V\overline{A}V$ where is the permutation matrix with units in the secondary diagonal.

Definition: 1 (2014)

A matrix $A \in H^{n\times n}$ is said to be quaternion hermitian doubly stochastic matrix if $A=A^{*}$ and $\sum_{i=1}^{n}|a_{ij}|=1$, $j=1,2,....\text{n}$ and $\sum_{j=1}^{n}|a_{ij}|=1$, $i=1,2,...n$ and $|a_{ij}|\geq 0$ if A is doubly stochastic and also quaternion hermitian then it is called a quaternion hermitian doubly stochastic matrix.

Example :1.1

$$A = \begin{bmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{bmatrix}$$

Definition : 2

A matrix $A \in H^{n\times n}$ is said to be $q-k$ - hermitian doubly stochastic[5] matrix if $\overline{A} = KA^{*}K$, where k is a permutation matrix associated with the permutation k(x).

$$\overline{A} = \begin{bmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j-k & -3+i+j+k & 6 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\overline{A} = K\,A^{*}K$$

Lemma :

For A is q – k hermitian doubly stochastic matrix then the following are equivalent.

(i) $\overline{A} = k\,A^{*}k$ and $A^{*} = k\,\overline{A}\,k$

(ii) $k\,A^{*} = KA$

(iii) $A^{*}K = AK$

(iv) $(KA)^{*} = A^{*}K$

(v) $(A^{*}K) = KA$

Example : 1.2

$$A = \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$\overline{A} = \begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j-k & -3+i+j+k & 6 \end{pmatrix}$$

$$A^{*} = \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$k=(1)(2\quad 3)\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

(i) $$kA^*k=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2+i+j+k&-2-i-j-k\\2-i-j-k&2&-3+i+j+k\\-2+i+j+k&-3-i-j-k&6\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$=\begin{pmatrix}1&2-i-j-k&-2+i+j+k\\2+i+j+k&2&-3-i-j-k\\-2-i-j-k&-3+i+j+k&6\end{pmatrix}=\overline{A}$$

$$k\overline{A}k=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2+i+j+k&-2-i-j-k\\2-i-j-k&2&-3+i+j+k\\-2+i+j+k&-3-i-j-k&6\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=A^*$$

(ii) $$kA^*=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2+i+j+k&-2-i-j-k\\2-i-j-k&2&-3+i+j+k\\-2+i+j+k&-3-i-j-k&6\end{pmatrix}$$

$$=\begin{pmatrix}1&2+i+j+k&-2-i-j-k\\-2+i+j+k&-3-i-j-k&6\\2-i-j-k&2&-3+i+j+k\end{pmatrix}=kA$$

(iii) $$A^*k=\begin{pmatrix}1&2+i+j+k&-2-i-j-k\\2-i-j-k&2&-3+i+j+k\\-2+i+j+k&-3-i-j-k&6\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&-2-i-j-k&2+i+j+k\\2-i-j-k&-3+i+j+k&2\\-2+i+j+k&6&-3-i-j-k\end{pmatrix}=AK$$

(iv) $$(K\,\mathrm{A})^*=\begin{pmatrix}1&-2-i-j-k&2+i+j+k\\2-i-j-k&-3+i+j+k&2\\-2+i+j+k&2&-3+i+j+k\end{pmatrix}=A^*K$$

(v) $$(\mathrm{A}^*K)^*=\begin{pmatrix}1&-2+i+j+k&-2-i-j-k\\-2+i+j+k&-3-i-j-k&6\\2-i-j-k&2&-3+i+j+k\end{pmatrix}=KA$$

Results :

Let $K\,A = \overline{AK}$ and $AK = \overline{KA}$

Theorem: 1

Let $A \in H^{n\times n}$ is $q-k$ - hermitiandoubly stochastic matrix then $\overline{A} = KA^*K$.

Proof:

$KA^*K = KAK$ where $KA^* = KA$.

$= \overline{AK}K$ where $KA = \overline{AK}$

$= \overline{AK}K = \overline{A}k^2$ where $\overline{K} = K$

$= \overline{A}$ where $K^2 = I$

Theorem : 2

Let $A \in H^{n\times n}$ is $q-k$ - hermitiandoubly stochastic matrix then $K\overline{A}K = A^*$

Proof:

$K\overline{A}K = K\overline{A}\overline{K}$ where $K = \overline{K}$

$= K\overline{AK} = KKA$ where $\overline{AK} = KA$

$= KKA^*$ where $KA = KA^*$

$= K^2A^* = A^*$ where $K^2 = I$

Theorem : 3

Let $A, B \in H^{n\times n}$ is $q-k$-hermitian doubly stochastic matrix then $\frac{1}{2}(A+B)$ is $q-k$-hermitian doubly stochastic matrix.

Proof:

Let A and B are $q-k$-hermitian doubly stochastic matrix if $\overline{A} = KA^*K$ and $\overline{B} = KB^*K$

To prove $\frac{1}{2}(A+B)$ is $q-k$-hermitian doubly stochastic matrix. We will show that $\frac{1}{2}\overline{(A+B)} = K\frac{1}{2}(A+B)^*K$.

Now $K\frac{1}{2}(A+B)^*K$

$= \frac{1}{2}K(A^*+B^*)K = \frac{1}{2}(KA^*+KB^*)K = \frac{1}{2}(KA^*K+KB^*K) = \frac{1}{2}(\overline{A}+\overline{B})$

$= \frac{1}{2}\overline{(A+B)}$ where $\overline{A} = KA^*K$ and $\overline{B} = KB^*K$

(ii) To prove $\frac{1}{2}(A+B)$ is $q-k$-hermitian doubly stochastic matrix. We will show that $\frac{1}{2}\overline{(\overline{A}+\overline{B})} = K\frac{1}{2}\overline{(\overline{A}+\overline{B})}^*K$

Now $K\frac{1}{2}(A+B)^*K = K\frac{1}{2}(A^*+B^*)K$

$= \frac{1}{2}(KA^*+KB^*)K = \frac{1}{2}(KA^*K+KB^*K)$

$= \frac{1}{2}\overline{(\overline{A}+\overline{B})} = \frac{1}{2}\overline{\overline{(A+B)}}$

$$=\frac{1}{2}\,(A+B)$$

$$\overline{A}=KA^{*}K \quad \overline{B}=KB^{*}K$$

Example : 1.3

$$A=\begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$\overline{A}=\begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i--k \\ -2-i-j-k & -3+i+j+k & 6 \end{pmatrix}$$

$$A^*=\begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$B=\begin{pmatrix} -2 & -3+2i-j+k & 6-2i+j-k \\ -3-2i+j-k & 6 & -2+2i-j+k \\ 6+2i-j+k & -2-2i+j-k & -3 \end{pmatrix}$$

$$\overline{B}=\begin{pmatrix} -2 & -3+2i+j-k & 6+2i-j+k \\ -3+2i-j+k & 6 & -2-2i+j-k \\ 6-2i+j-k & -2+2i-j+k & -3 \end{pmatrix}$$

$$B^*=\begin{pmatrix} -2 & -3+2i-j+k & 6-2i+j-k \\ -3-2i+j-k & 6 & -2+2i-j+k \\ 6+2i-j+k & -2-2i+j-k & -3 \end{pmatrix}$$

$$(A+B)=\begin{pmatrix} -1 & 1+3i+2k & 4-3i-2k \\ -1-3i-2k & 8 & -5+3i+2k \\ 4+3i+2k & -5-3i-2k & 3 \end{pmatrix}$$

Here (A+B) is also quaternion hermitian matrix.

$$\tfrac{1}{2}(A+B)=\begin{pmatrix} -\tfrac{1}{2} & \tfrac{1}{2}+\tfrac{3}{2}\,i+k & 2-\tfrac{3}{2}\,i-k \\ -\tfrac{1}{2}-\tfrac{3}{2}\,i-k & 4 & -\tfrac{5}{2}+\tfrac{3}{2}i+k \\ 2+\tfrac{3}{2}\,i+k & -\tfrac{5}{2}-\tfrac{3}{2}i-k & \tfrac{3}{2} \end{pmatrix}$$

$$\tfrac{1}{2}(\overline{A+B}) = \begin{pmatrix} -\tfrac{1}{2} & -\tfrac{1}{2}-\tfrac{3}{2}\,i-k & 2+\tfrac{3}{2}\,i+k \\ \tfrac{1}{2}+\tfrac{3}{2}\,i+k & 4 & -\tfrac{5}{2}-\tfrac{3}{2}i-k \\ 2-\tfrac{3}{2}\,i-k & -\tfrac{5}{2}+\tfrac{3}{2}i+k & \tfrac{3}{2} \end{pmatrix}$$

$$\left(A+B\right)^* = \begin{pmatrix} -1 & -1-3\,i-2k & 4+3\,i+2k \\ 1+3\,i+2k & 8 & -5-3i-2k \\ 4-3\,i-2k & -5+3i+2k & 3 \end{pmatrix}$$

$$1/2\left(A+B\right)^* == \begin{pmatrix} -\tfrac{1}{2} & -\tfrac{1}{2}-\tfrac{3}{2}\,i-k & 2+\tfrac{3}{2}i+k \\ \tfrac{1}{2}+\tfrac{3}{2}i+k & 4 & -\tfrac{5}{2}-\tfrac{3}{2}i+k \\ 2-\tfrac{3}{2}i-k & -\tfrac{5}{2}+\tfrac{3}{2}i+k & \tfrac{3}{2} \end{pmatrix}$$

$$1/2\,K\left(A+B\right)^* K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\tfrac{1}{2} & \tfrac{1}{2}+\tfrac{3}{2}\,i+k & 2-\tfrac{3}{2}i-k \\ \tfrac{1}{2}-\tfrac{3}{2}\,i-k & 4 & -\tfrac{5}{2}+\tfrac{3}{2}i+k \\ 2+\tfrac{3}{2}\,i+k & -\tfrac{5}{2}-\tfrac{3}{2}i-k & \tfrac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\tfrac{1}{2} & \tfrac{1}{2}+\tfrac{3}{2}\,i+k & 2-\tfrac{3}{2}i-k \\ 2+\tfrac{3}{2}i+k & -\tfrac{5}{2}-\tfrac{3}{2}i-k & \tfrac{3}{2} \\ \tfrac{1}{2}-\tfrac{3}{2}i+k & 4 & -\tfrac{5}{2}+\tfrac{3}{2}i+k \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\tfrac{1}{2} & \tfrac{1}{2}+\tfrac{3}{2}\,i+k & 2-\tfrac{3}{2}i-k \\ \tfrac{1}{2}-\tfrac{3}{2}\,i-k & 4 & -\tfrac{5}{2}+\tfrac{3}{2}i+k \\ 2+\tfrac{3}{2}\,i+k & -\tfrac{5}{2}-\tfrac{3}{2}i-k & \tfrac{3}{2} \end{pmatrix}$$

Theorem : 4

If A and B are $q-k$-hermitian doubly stochastic matrix then AB is not an $q-k$-hermitian doubly stochastic matrix.

Proof:

Let A and B are $q-k$-hermitian doubly stochastic matrix if $\overline{A} = KA^*K$ and $\overline{B} = KB^*K$

Since A^* and B^* are also $q-k$-hermitian doubly stochastic matrices then $A^* = K\overline{A}K$ and $B^* = K\overline{B}K$
$AB = (KA^*K)(KB^*K) = KA^*B^*K$

To prove AB is not an $q-k$-hermitian doubly stochastic matrix we will show that $AB \neq \overline{BA}$

Now $K(AB)^* K \neq K(B^* A^*)K$

$= K(K\overline{B}K)(K\overline{A}K)K$ where

$A^* = K\overline{A}K$ and $B^* = K\overline{B}K$

$= \mathrm{K}^2\overline{\mathrm{B}}\ \mathrm{K}^2\overline{\mathrm{A}}\ \mathrm{K}^2$ Where $K^2 = I$

$= \overline{BA} \neq AB$

therefore quaternion matrices does not satisfy commute

Hence $AB \neq BA$.

Example : 1.4

$$A = \begin{pmatrix} 1 & 2+i+j+k & -2-i-k \\ 2+i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} -2 & -3+2i-j+k & 6-2i+i-k \\ -3-2i+j-k & 6 & -2+2i-j+k \\ 6+2i-j+k & -2-2i-j-k & -3 \end{pmatrix}$$

$$AB = \begin{pmatrix} -16 & 11+8i+5j+7k & 6+i+4j+2k \\ -30 & 16 & 19+i-5j-k \\ 10i-8j+4k+49 & -18i-26 & -21 \end{pmatrix}$$

Here AB is not an quaternion hermitian doubly stochastic matrix.[5]

Definition : 3

A matrix $\mathrm{A} \in H^{n\times n}$ is $q-s$ - hermitian doubly stochastic matrix if $\overline{A} = VA^*V$ where V is a permutation matrix with units in the secondary diagonal.

Lemma :

For A is $q-s$ - hermitian doubly stochastic matrix then the following are equivalent.

i. $\overline{A} = VA^*V$ and $A^* = V\overline{A}V$

ii. $VA^* = VA$

iii. $A^*V = AV$

iv. $(VA)^* = A^*V$

v. $(A^*V)^* = VA$

Example : 1.5

$$A = \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$\overline{A} = \begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j+-k & -3+i+j+k & 6 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & 2+i+j=k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) $$VA^*V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j-k & -3+i+j+k & 6 \end{pmatrix} = \overline{A}$$

$$V\overline{A}V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j-k & -3+i+j+k & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j-k & -3+i+j+k & 6 \end{pmatrix} = A^*$$

(ii) $$VA^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2-i-j-k & 2+i+j+k & 1 \\ -3+i+j+k & 2 & 2-i-j-k \\ 6 & -3-i-j-k & -2+i+j+k \end{pmatrix} = VA$$

(iii) $$A^*V = \begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2-i-j-k & 2+i+j+k & 1 \\ -3+i+j+k & 2 & 2-i-j-k \\ 6 & -3-i-j-k & -2+i+j+k \end{pmatrix} = AV$$

(iv) $$(VA)^* = \begin{pmatrix} -2-i-j-k & 2+i+j+k & 1 \\ -3+i+j+k & 2 & 2-i-j-k \\ 6 & -3-i-j-k & -2+i+j+k \end{pmatrix} = A^*V$$

(v) $$(A^*V)^* = \begin{pmatrix} -2-i-j-k & 2+i+j+k & 1 \\ -3+i+j+k & 2 & 2-i-j-k \\ 6 & -3-i-j-k & -2+i+j+k \end{pmatrix} = VA$$

Results: $VA = \overline{AV}$ and $AV = \overline{VA}$.

Theorem : 5

Let $A \in H^{n\times n}$ is $q-s$ - hermitian doubly stochastic matrix then $\overline{A} = VA^*V$

Proof:

$VA^*V = VAV$ where $VA^* = VA$

$= \overline{A}VV$ Where $VA = \overline{AV}$

$= \overline{A}\overline{V}V = \overline{A}V^2$ Where $\overline{K} = K$

$= \overline{A}$ where $V^2 = I$

Theorem : 6

Let $A \in H^{n\times n}$ is $q-s$ - hermitian doubly stochastic matrix if $A^* = V\overline{A}V$

Proof :

$V\overline{A}V = V\overline{A}V$ Where $V = \overline{V}$

$= V\overline{A}V = VVA$ Where $\overline{AV} = VA$

$= VVA^*$ where $VA = VA^*$

$= V^2A^*$ where $V^2 = I$

Theorem : 7

Let $A, B \in H^{n\times n}$ is $q-s$ -hermitian doubly stochastic matrix then $\frac{1}{2}(A+B)$ is $q-s$ -hermitian doubly stochastic matrix

Proof :

Let A and B ae $q-s$ - hermitian doubly stochastic matrix if $\overline{A} = VA^*V$ $\overline{B} = VB^*V$

To prove $\frac{1}{2}(A+B)$ is $q-s$ -hermitian doubly stochastic matrix. We will show that $\frac{1}{2}(\overline{A+B}) = V\frac{1}{2}(A+B)^*V$

Now $V\frac{1}{2}(A+B)^*V = \frac{1}{2}(A^*+B^*)V$

$$= \frac{1}{2}(VA^*+VB^*)V$$

$$=\frac{1}{2}(VA^{*}V+VB^{*}V)$$

$\frac{1}{2}(\overline{A}+\overline{B})=\frac{1}{2}(\overline{A+B})$ where $\overline{A}=VA^{*}V$ $\overline{B}=VB^{*}V$

Theorem : 8

If A and B are $q-s$ - hermitian doubly stochastic matrix the AB is not an $q-s$ Hermitian doubly stochastic matrix.

Proof :

Let A and B ae $q-s$ - hermitian doubly stochastic matrix if $\overline{A}=VA^{*}V$ and $\overline{B}=VB^{*}V$

Since A^{*} and B^{*} are also s-hermitian doubly stochastic matrices then $A^{*}=V\overline{A}V$ and $B^{*}V\overline{B}V$

To prove AB is not an $q-s$ - hermitian doubly stochastic matrices, we will show that $AB\neq\overline{BA}=V(AB)^{*}V$

Now $V(AB)^{*}V=V(B^{*}A^{*})V=V(V\overline{B}V)(V\overline{A}V)V$ Where $A^{*}=V\overline{A}V$ and $B^{*}V\overline{B}V$

$=V^{2}\overline{B}V^{2}\overline{A}V^{2}=\overline{B}\overline{A}$ where $V^{2}=I$.

Hence $AB\neq\overline{BA}$.

Definition : 4

A matrix $A\in H^{n\times n}$ is said to be $q-s-k$ -hermitian doubly stochastic matrix if.

(i) $A=KVA^{*}VK$

(ii) $\overline{A}=KV\overline{A}VK$

(iii) $A=VKA^{*}KV$

(iv) $\overline{A}=VK\overline{A}KV$

'k' is a permutation matrix and $k(x)=(1)(2\ \ 3)$

Theorem : 9

Let $A\in H^{n\times n}$ is said to be $q-s-k$ -hermitian doubly stochastic matrix then

(i) $A^{*}=KVA^{*}VK$

(ii) $\overline{A}=KV\overline{A}VK$

(iii) $A^{*}=VKA^{*}KV$

(iv) $\overline{A}=VK\overline{A}KV$

Proof :

$KVA*VK=K(VA*V)K=K\overline{A}K$ Where $VA*V=\overline{A}=A*$ Where $K\overline{A}K=A*$

$KV\overline{A}*VK=K(V\overline{A}*V)K=KA*K$ Where $V\overline{A}V=A*=\overline{A}$ Where $KA*K=\overline{A}$

$VKA*KV=V(KA*K)V=V\overline{A}V$ Where $KA*K=\overline{A}=A*$ Where $V\overline{A}V=A*$

$VK\overline{A}KV=V(K\overline{A}K)V=VA*V$ Where $K\overline{A}K=A*=\overline{A}$ Where $VA*V=\overline{A}$

Example : 1.6

$$A=\begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$\overline{A}=\begin{pmatrix} 1 & 2-i-j-k & -2+i+j+k \\ 2+i+j+k & 2 & -3-i-j-k \\ -2-i-j-k & -3+i+j+k & 6 \end{pmatrix}$$

$$A^*=\begin{pmatrix} 1 & 2+i+j+k & -2-i-j-k \\ 2-i-j-k & 2 & -3+i+j+k \\ -2+i+j+k & -3-i-j-k & 6 \end{pmatrix}$$

$$K=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad V=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad KV=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} VK=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Theorem : 10

Let $A,B\in H^{n\times n}$ is $q-s-k$ -hermitian doubly stochastic matrix then $\frac{1}{2}(A+B)$ is $q-s-k$ hermitian doubly stochastic matrix.

Proof :

Let A and B are $q-s-k$ -hermitiandoubly stochastic matrix if $A^*=KVA^*VK$ and $B^*=KVB^*VK$. To prove $\frac{1}{2}(A+B)$ is $q-s-k$ -hermitian doubly stochastic matrix. We will show that $\frac{1}{2}(A+B)^* = KV\frac{1}{2}(A+B)^*VK$

Now $KV\frac{1}{2}(A+B)^*VK = K(V\frac{1}{2}(A+B)^*V)K$

$=K\frac{1}{2}(\overline{A+B})\mathrm{K}$ (Using theorem 7)

$= \frac{1}{2}(A+B)^*$ using theorem (3)

Theorem:11

If A and B are $q-s-k$ -hermitian doubly stochastic matrix then AB is not an $q-s-k$ Hermitian doubly stochastic matrix.

Proof :

Let A and B are $q-s-k$ -hermitian doubly stochastic matrix if $A^*=KVA^*VK$ and $B^*=KVB^*VK$

To prove AB is not $q-s-k$ -hermitian doubly stochastic matrix. We will show that $(AB)^{*} \neq KV(AB)^{*}\mathrm{VK}$.

Now $\mathrm{KV(AB)}^{*}VK \neq K(V(AB)^{*})VK \neq K(\overline{BA})K$ (Using theoem 8)

$= \mathrm{(AB)}^{*}$ (Using theorem 4)

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