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A study on limit cycle and non-homoclinic orbits for FitzHugh-Nagumo System

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Abstract

In this paper we investigate the complete FitzHugh-Nagumo System with $I \neq 0$. Based on the result in[1,2] we discuss the non-existence of homoclinic orbits of the system. Further, we prove that the system has unique limit cycle under the conditions of existence of homoclinic orbits.

Keywords : Lie'nard equation, FitzHugh-Nagumo System, Limit cycle, Homoclinic orbits.
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1 Introduction

In the present paper, we revisit the problem of existence and uniqueness of limit cycle. We give criterion for the model (FitzHugh-Nagumo System) to have or not Homoclinic orbits with $I \neq 0$.

Now, we consider the following Lie'nard system

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x). \end{aligned} \tag{1.1}$$

The main part of this paper is devoted to explain the non-existence of Homoclinic orbits and uniqueness of limit cycles of FitzHugh-Nagumo System as given by the following differential system.

$$\begin{aligned} \dot{x} &= y - Ax(x-B)(x-\lambda) + I, \\ \dot{y} &= \varepsilon(x - \delta y). \end{aligned} \tag{1.2}$$

We investigate the System with the parameters $A, B, \delta, \varepsilon, \lambda, I$ being not zeros. In particular, we study the system under the case $A = B = 1$ and $\lambda \neq 1$, where $\delta \in (-1,0)$, $I \in R.$

This system has been extensively studied with particular emphasis on bifurcate limit cycles as well as in modeling of certain phenomenon. From literature review, it is noticed that, most of the articles studied the system under some parameters being zeros, for instance see[4,5,6,7,9,10,11]. Luo Ding Jun in[8] investigated the particular of case $(1+\lambda)=0$, and proved the uniqueness of limit cycle. In[12] there is a general analysis of the system for bifurcation of limit cycles from Hopf-bifurcation. In[13], we studied the system (1.2) with all parameters not zeros and proved the uniqueness of limit cycle. There are many articles in the field of limit cycles and homoclinic orbits for example see[20,21,22].

The main focus of this present paper is to consider existence of homoclinic orbits of the system for two different cases, and through these cases we discuss the existence and the uniqueness of limit cycle. Note, that there are some results for non-existence of homoclinic orbits in the case $1+\lambda=0$ e.g. see[3].

In order to study the existence and non-existence of limit cycles and homoclinic orbits, we make change of variables to get Lie'nard type (1.1). Let $x-\alpha \to x$ and $y+\delta\varepsilon x+\frac{\alpha}{\delta}\to y$ where α is the root of the equilibrium equation $\delta x^3-\delta(1+\lambda)x^2-(\delta\lambda-1)x-I\delta=0.$ Then system (1.2) becomes,

$$\begin{aligned}\dot{x} &= y-[x^3+(3\alpha-(1+\lambda))x^2+(3\alpha^2-2(1+\lambda)\alpha+\lambda+\delta\varepsilon)x],\\ \dot{y} &= -\delta\varepsilon[x^3+(3\alpha-(1+\lambda))x^2+(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta})x].\end{aligned}\tag{1.3}$$

The paper is organized as follows.
In section 2, we use some of the existing theorems and lemmas to obtain specific results.
In section 3, we investigate the existence and uniqueness of limit cycle system which has homoclinic orbits.
In section 4, we study the existence of homoclinic orbits and prove that system has no homoclinic orbits.

In section 5, the existence and uniqueness of critical point $O(0,0)$ and its global asymptotic stability are discussed for the system (1.2).

A note on special case for $1+\lambda=0$ is given in section 6. The conclusion of this study is given in section 7.

2. Methods and Tools: Some useful Theorems and Lemmas :

For the existence of limit cycle let us consider the method of amplitude of limit cycles. Consider the following system

$$\begin{aligned}\dot{x} &= y+\varepsilon f(x),\\ \dot{y} &= -x+\varepsilon g(x),\end{aligned}\tag{2.1}$$

then the amplitude of limit cycles is given by

$$\beta(a)=\int_{-a}^{a}[\tilde{g}(x,\sqrt{a^2-x^2})+\frac{x\tilde{f}(x,\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}]dx=0,\tag{2.2}$$

where

$$\begin{aligned}\tilde{f}(x,y) &\equiv \frac{1}{2}[f(x,y)+f(x,-y)-f(-x,y)-f(-x,-y)],\\ \tilde{g}(x,y) &\equiv \frac{1}{2}[g(x,y)-g(x,-y)+g(-x,y)-g(-x,-y)].\end{aligned}\tag{2.3}$$

The positive solution of $\beta(a)=0$ is the amplitude of limit cycle of (2.1). Observe that $\tilde{f}(x,y)$ is an odd function of x and an even function of y, whereas $\tilde{g}(x,y)$ is an even function of x and an odd function of y. For more details see[20].

To prove the uniqueness of limit cycle we can apply the following theorem:

Theorem 2.1[13] : Let $f(x)$, $g_1(x)$ and $k(x)$ be continuous functions on (x_2,x_1), where $-\infty\le x_2<0<x_1\le+\infty$. Suppose the system (1.1) satisfies the following conditions:

(1) $xg_1(x) > 0$, $x \in (x_2, x_1)$ and $x \neq 0$.

(2) $k(x) > 0$ for all $x \in (x_2, x_1)$, or $k(0) = 0$, $k(x) > 0$ for $x \in (x_2, 0) \cup (0, x_1)$.

(3) $\frac{f(x)}{g(x)}$ is increasing for $x \in (x_2, 0) \cup (0, x_1)$, and $\frac{f(x)}{g(x)} \neq 0$ in a neighborhood of the origin.

Then the system (1.1) has at most one limit cycle in the strip

$D := (x, y) \mid x_2 < x < x_1, -\infty < y < +\infty$.

Moreover, the limit cycle is stable if it exists.

Here $g(x) = g_1(x)k(x)$ and $f(x) = F'(x)$

For the existence of homoclinic orbits let us consider the following system

$$\begin{aligned} \dot{x} &= a(y) + b(x), \\ \dot{y} &= c(y) + d(x). \end{aligned} \tag{2.4}$$

Lemma 2.2[1] If there exists a homoclinic orbit in any neighborhood of the origin for (2.4), then $b'(0) = c'(0) = 0$, and Det J(0, 0) = 0.

Therefore, by this lemma we have $F(0) = F'(0) = 0$ see also Proposition 2.7 in[1]. Thus in the sequel we study the system (1.3) through two different cases

1. $3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon = 0$;
2. $\alpha = 0$ and $\lambda + \delta\varepsilon = 0$.

The first three focal values of system (1.3) are (see Lemma 3.3.1 in[8]);

$$\begin{aligned} W_1 &= -\frac{(3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon)}{\sqrt{\mu}}, \\ W_2 &= \frac{\delta\varepsilon[2(3\alpha - (1+\lambda)^2 - 3(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta})]}{8\mu^2\sqrt{\mu}}, \\ W_3 &= -\frac{15c\delta\varepsilon}{\mu^3\sqrt{\mu}}, \end{aligned} \tag{2.5}$$

such that $\mu = \delta\varepsilon(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta})$.

3 The uniqueness of limit cycle :

In this section, we investigate the uniqueness of limit cycle for the system (1.3) under above two cases as given in section 2.

Case:1 $3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon = 0$

Therefore system (1.3) becomes,

$$\begin{aligned} \dot{x} &= y - [x^3 + (3\alpha - (1+\lambda))x^2], \\ \dot{y} &= -\delta\varepsilon[x^3 + (3\alpha - (1+\lambda))x^2 + (3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta})x]. \end{aligned} \tag{3.1}$$

We note that $F(0) = 0$, $g(0) = 0$. The other roots of $F(x) = 0$ and $g(x) = 0$ respectively are

$$x = -(3\alpha - (1+\lambda)), \tag{3.2}$$

$$x=\frac{1}{2}[-(3\alpha-(1+\lambda))\pm\sqrt{(3\alpha-(1+\lambda))^2-4(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta})}]. \qquad (3.3)$$

Therefore the system has unique singular point for $3\alpha-(1+\lambda))^2-4(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta}<0,$ and for $3\alpha-(1+\lambda))^2-4(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta}>0$ the system (3.1) has three singular points.

Since we consider the system in the case O as anti-saddle then we have $\lambda-\frac{1}{\delta}>0,\ (1+\lambda)^2-3(\lambda-\frac{1}{\delta})<0$ and $\delta\varepsilon+\frac{1}{\delta}<0.$ So, from these values we can find $\Delta=(3\alpha-(1+\lambda))^2-4(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta})<0.$ Thus, we deduces that the system (3.1) has $O(0,0)$ as unique critical point. The graphs of $F(x)$ and $g(x)$ in Figure 1 gives better understanding.

Let us apply equation (2.2) to get the conditions of the existence of limit cycles:

$$\beta(a)=\pi a^3(\frac{2(3\alpha-(1+\lambda))}{3\mu}+\frac{3a}{8\mu\sqrt{\mu}}),$$

where $\mu=\delta\varepsilon(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta})>0.$

Therefore $a=-\frac{16\sqrt{\mu}}{9}(3\alpha-(1+\lambda)),$ so we get the amplitude of limit cycle only in case when $(3\alpha-(1+\lambda))<0.$

From (2.5) we have

$$\begin{aligned}W_1 &= 0,\\ W_2 &= \frac{\delta\varepsilon[2(3\alpha-(1+\lambda))^2+3(\delta\varepsilon+\frac{1}{\delta})]}{8\mu^2\sqrt{\mu}},\\ W_3 &= -\frac{15c\delta\varepsilon}{\mu^3\sqrt{\mu}}<0.\end{aligned} \qquad (3.4)$$

If $(\delta\varepsilon+\frac{1}{\delta})>0$ then the system has $O(0,0)$ as saddle point therefore no limit cycle. In the case of $(\delta\varepsilon+\frac{1}{\delta})<0$ and the value $2(3\alpha-(1+\lambda))^2+3(\delta\varepsilon+\frac{1}{\delta})>0,$ one limit cycle appear from Hopf-bifurcation. Thus from above discussion we have the following theorem.

Theorem 3.1 For $3\alpha-(1+\lambda)<0$ and $2(3\alpha-(1+\lambda))^2+3(\delta\varepsilon+\frac{1}{\delta})>0,$ the system (3.1) has unique limit cycle.

Proof: Now, we apply Theorem 2.1. Let $g_1(x)=x,$ since $\Delta<0$ and $(3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta})>0$ then we have $K(x)=\delta\varepsilon[x^3+(3\alpha-(1+\lambda))x+3\alpha^2-2(1+\lambda)\alpha+\lambda-\frac{1}{\delta}]>0$ for all $x\in R.$ For the condition $\frac{f(x)}{g_1(x)}$ increasing we have $(\frac{f(x)}{g_1(x)})'=3>0$ for $x\in R$ Thus, from Lemma 1.1 the system (2.3) with $b>0$ and $c>0$ has at most one stable limit cycle.

Case : 2 $\alpha = 0$ **and** $\lambda + \delta\varepsilon = 0$
In this case equation (1.3) becomes,

$$\begin{aligned} \dot{x} &= y - [x^3 - (1+\lambda)x^2] \\ \dot{y} &= -\delta\varepsilon[x^3 - (1+\lambda)x^2 + (\lambda - \frac{1}{\delta})x]. \end{aligned} \tag{3.5}$$

Here $\lambda = -\delta\varepsilon < 0$, and since we assume that O is anti-saddle then $(\lambda - \frac{1}{\delta}) > 0$. The discriminate of $g(x)$ is $\Delta = (1+\lambda)^2 - 4(\lambda - \frac{1}{\delta})$ suppose that $\Delta \geq 0$ then we find that $\lambda \geq 3$ contradiction since $\lambda < 0$. Thus $g(x)$ has unique critical point is $O(0,0)$. If we apply the equation (2.2), and as in the previous case, we find that $(1+\lambda) > 0$ is the condition of existence of limit cycle.
The focal values of (3.5) are

$$\begin{aligned} W_1 &= 0, \\ W_2 &= \frac{\delta\varepsilon[2(1+\lambda)^2 + 3(\delta\varepsilon + \frac{1}{\delta})]}{8\mu^2\sqrt{\mu}}, \\ W_3 &= -\frac{15c\delta\varepsilon}{\mu^3\sqrt{\mu}} < 0. \end{aligned} \tag{3.6}$$

In a similar way one can obtain Theorem 3.1 with following result:

Theorem 3.2 For $(1+\lambda) > 0$ and $2(1+\lambda)^2 + 3(\delta\varepsilon + \frac{1}{\delta}) > 0$ the system (3.5) has unique limit cycle.

4 The existence of homoclinic orbits :

In this section we study the existence of homoclinic orbit through the above two cases as given in section 2. The result can be obtained through the following lemma:

Lemma 4.1[2] Suppose there exists a $\sigma > 0$ such that $F(x) > 0$ for $0 < |x| < \sigma$. If

$$\frac{1}{F(x)} \int_0^x \frac{g(\xi)}{F(\xi)} d\xi > \frac{1}{4} \tag{4.1}$$

for $0 < x < \sigma$ are satisfied, then the system (1.3) with $3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon = 0$ has no homoclinic orbits.
Let

$$\sigma = \frac{-\eta + \sqrt{\eta^2 + 12k\delta\varepsilon}}{2}$$

where $\eta = (3\alpha - (1+\lambda))$ and k is a constant such that

$$0 < k < min\{\frac{2\eta^2}{3\delta\varepsilon}, 1\}$$

so, we have $0 < \sigma < \eta$. Since $3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta} > 0$, we obtain

$$\begin{aligned} \frac{1}{F(x)} \int_0^x \frac{g(\xi)}{F(\xi)} d\xi &> \frac{\delta\varepsilon}{x^2(x+\eta)} \int_0^x d\xi \\ &= \frac{\delta\varepsilon}{x(x+\eta)} \\ &> \frac{k\delta\varepsilon}{x(x+\eta)} > \frac{1}{4} \end{aligned} \tag{4.2}$$

for $0 < \sigma < \eta$. Thus, we deduced that the system (1.3) with $3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon = 0$ has no homoclinic orbits.

As a proof of this lemma we can get the following result

Lemma 4.2 For $\alpha = 0$ and $\lambda + \delta\varepsilon = 0$ the system (1.3) has no homoclinic orbits.

Remark 4.3 It is important to point out that, we consider the following two different cases such as $3\alpha - (1+\lambda) > 0$ and $(1+\lambda) < 0$ in which the limit cycle does not exists.

5 Non-existence of limit cycle :

In this section we investigate two different cases of non-existence of limit cycle and global asymptotic stability of the critical point $O(0,0)$ which mainly depends on the above two sections.

Definition The equilibrium point $x = O$ is asymptotically stable iff:

1. $x = O$ is stable equilibrium point;
2. $\forall t_0 \geq 0\ \exists \delta(t_0)$ s.t $|x(t_0)| < \delta \Rightarrow lim_{\tau\to\infty} |x(\tau)| = O$

Now, we apply the focal values to the systems under the cases of non-existence of homoclinic orbits.

Consider the cases $3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon = 0$ and $(\delta\varepsilon + \frac{1}{\delta}) < 0$, and from equation (3.4) if $2(3\alpha - (1+\lambda))^2 + 3(\delta\varepsilon + \frac{1}{\delta}) < 0$, then no change of stability so no limit cycle and $O(0,0)$ is stable focus. Thus from this discussion and Lemma 4.1, we deduce that the unique critical point $O(0,0)$ is globally asymptotically stable.

Consider the cases $\alpha = 0$ and $\lambda + \delta\varepsilon = 0$ and $(\delta\varepsilon + \frac{1}{\delta}) < 0$.

As before and by using equation (3.6) and Lemma 4.2, we obtain that for $(\delta\varepsilon + \frac{1}{\delta}) < 0$ and $2(1+\lambda)^2 + 3(\delta\varepsilon + \frac{1}{\delta}) < 0$ the unique critical point $O(0,0)$ is globally asymptotically stable.

Remark 5.1 Consider $2(3\alpha - (1+\lambda))^2 + 3(\delta\varepsilon + \frac{1}{\delta}) < 0$ as condition 1 (c_1),

$2(1+\lambda)^2 + 3(\delta\varepsilon + \frac{1}{\delta}) < 0$ as condition 2 (c_2), then we get the following result

Theorem 5.2 Assume the condition c_1 or condition c_2 is satisfied. Then the equilibrium point $O(0,0)$ of system (3.1) or of system (3.5) is globally asymptotically stable respectively see.

6 A Note on Special Case $1+\lambda = 0$*:*

Consider system (3.1), and if we apply equations (2.2) and (3.4) then we have the following lemma

Lemma 6.1 For $\alpha < 0$ and $6\alpha^2 + (\delta\varepsilon + \frac{1}{\delta}) > 0$, then system (3.1) has unique limit cycle. If $\alpha < 0$ and $6\alpha^2 + (\delta\varepsilon + \frac{1}{\delta}) < 0$, then the equilibrium point $O(0,0)$ of system (3.1) is globally asymptotically stable

For system (3.5), and if $(1 + \frac{1}{\delta}) > 0$ then the system has $O(0,0)$ as a saddle, and for $(1 + \frac{1}{\delta}) < 0$ we have the following result $F(x)g(x) = \delta\varepsilon x^4[x^2 - (1 + \frac{1}{\delta})] > 0$ so no limit cycle. Thus from equation (3.6) we have:

Lemma 6.2 For $(1 + \frac{1}{\delta}) < 0$ the equilibrium point $O(0,0)$ of system (3.5) is globally asymptotically stable.

For more details see [3] and [8].

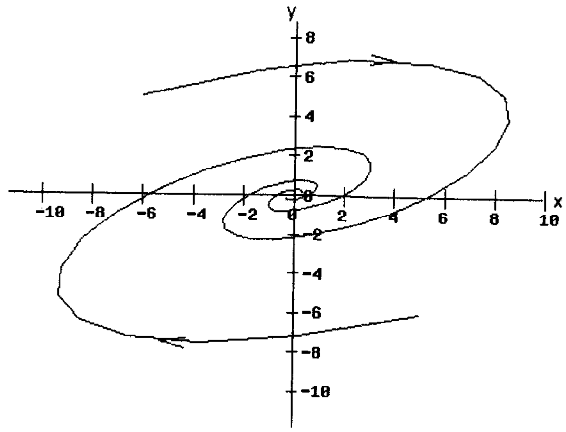

Figure 1. Stable Focus

7 Conclusion

The main goal of this paper is to investigate the complete FitzHugh-Nagumo System with $I \neq 0$ including the non- existence of homoclinic orbits of the system. Further, it is proved that the system has unique limit cycle under certain conditions of existence of homoclinic orbits.

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