ISSN 2231-346X (Print)
JUSPS-A Vol. 29(4), 169-175 (2017). Periodicity-Monthly
(Online) ISSN 2319-8044
Section A
JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

Lipschitz Condition in the Controlled Convergence Theorem

SANDY MAE S. DOCDOC[1] and JULIUS V. BENITEZ[2]

Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Lanao del Norte, Philippines

Corresponding Author Email: [2]julius.benitez@g.msuiit.edu.ph
Co-Author Email: [1]sundaymay.docdoc@gmail.com

http://dx.doi.org/10.22147/jusps-A/290406

Acceptance Date 15th March, 2017 **Online Publication Date 2nd April, 2017**

Abstract

The Lebesgue integral is noted for its powerful convergence theorems - the Monotone Convergence Theorem (MCT) and Dominated Convergence Theorem (DCT). In [5] and [8], these two convergence theorems were proved for the Henstock integral. Nakanishi in [9] and Lee and Yyborny in [8] consider yet another but more powerful convergence theorem, called the Controlled Convergence Theorem (CCT), that includes the monotone and dominated convergence theorems. Paredes and Chew in [11] studied a controlled convergence theorem for Banach space valued HL-integrals. Generalized absolute continuity (ACG) plays a very significant role in CCT. On the other hand, it is known that if a function satisfies a Lipschitz condition then it is ACG. It is the objective of this study to investigate some Lipschitz condition in the Controlled Convergence Theorem.

2010 Mathematics Subject Classification: 26A24, 26A06, 26A39, 26A42

Keywords: uniform δ-Lipschitz, $UACG^*$, Controlled Convergence Theorem

1 Introduction

One of the well-known, if not the strongest, convergence theorem of the Henstock-Kurzweil integral is the Controlled Convergence Theorem (CCT). In [2], Kurzweil and Jarǹk give a version of this theorem for Perron-type integrals. For real-valued functions, this convergence theorem was considered by Lee and Chew in their papers [6] and [7] and in the books [5] and [8]. A version of CCT for functions taking values in fuzzy numbers was also considered in [3] and [13]. CCT for Banach-valued functions defined on a compact interval $[a, b] \subseteq \mathbb{R}$ and strong variational Banach-valued multiple integrals were studied by Paredes, et.al. in [11] and [12], respectively. Several versions of CCT were also given by Nakanishi in [9], S. Pal, D.K. Ganguly and L.P. Yee in [10], and Wang in [14].

Controlled convergence theorem is formulated using a special version of "generalized continuity" called ACG^*. On the other hand, Lipschitz conditions, which are simpler but stronger conditions compared to continuity, are constantly used in the theory of differential equations. In this paper, we give some relationships between uniform ACG^* and concepts satisfying some uniform Lipschitz conditions. In conclusion, a version of the controlled convergence theorem is formulated.

2 Basic Concepts

A *gauge* on $[a, b]$ is a positive function $\delta(x)$ on $[a, b]$. A *Henstock δ-fine division* of $[a, b]$ is a finite collection $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ of non-overlapping interval-point pairs such that for all $i = 1, 2, \ldots, n$

$$\xi_i \in [x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \text{ and } \bigcup_{i=1}^n [x_{i-1}, x_i] = [a, b].$$

We say that $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is a *McShane δ-fine division* of $[a, b]$ if for all $i = 1, 2, \ldots, n$

$$[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)),\ \xi_i \in [a, b], \quad \text{and} \quad \bigcup_{i=1}^n [x_{i-1}, x_i] = [a, b].$$

This means that every Henstock δ-fine divisions of $[a, b]$ are McShane δ-fine.

Lemma 2.1 (Cousin's Lemma) [8] *If $\delta(x) > 0$ is a gauge on $[a, b]$, then there exists a δ-fine division of $[a, b]$.*

Definition 2.2 [8]A function $f : [a, b] \to \mathbb{R}$ is said to be *Henstock integrable* to A on $[a, b]$ if for each $\epsilon > 0$, there exists a gauge $\delta(\xi) > 0$ on $[a, b]$ such that whenever $D = \{([u, v], \xi)\}$ is a δ-fine division of $[a, b]$, we have

$$\left|(D) \sum f(\xi)(v - u) - A\right| < \epsilon.$$

If $f : [a, b] \to \mathbb{R}$ is Henstock integrable to A, we write

$$A = (\mathcal{H}) \int_a^b f(x) dx,$$

and call A as the *Henstock integral* of f on $[a, b]$. The function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = (\mathcal{H}) \int_a^x f(t)\ dt.$$

is called a *primitive* of f on $[a, b]$.

Definition 2.3 [1] Let $E \subseteq \mathbb{R}$. A function $F : E \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on E or simply *Lipschitz* on E if there exists $L > 0$ such that

$$|F(y) - F(x)| \leq L \cdot |y - x|$$

for each $x, y \in E$. The number L is called a *Lipschitz constant.*

3 Uniform Lipschitz

Definition 3.1 A sequence $\langle F_n \rangle_{n=1}^{\infty}$ of real-valued functions defined on $[a, b]$ is said to be ***uniformly Lipschitz on*** $[a, b]$ if there exists $L > 0$ such that for each $n \in \mathbb{N}$,

$$|F_n(v) - F_n(u)| \leq L \cdot |v - u|, \text{ for each } u, v \in [a, b].$$

Theorem 3.2 *Let $\langle F_n \rangle_{n=1}^{\infty}$ be a sequence of functions defined on $[a, b]$. Then the following are equivalent:*

(*i*) *$\langle F_n \rangle_{n=1}^{\infty}$ is uniformly Lipschitz on $[a, b]$.*

(*ii*) *For each $x \in [a, b]$, there exists $L(x) > 0$ and $\delta(x) > 0$ such that for every McShane δ-fine interval-point pair $([u, v], x)$ and for each $n \in \mathbb{N}$, we have*

$$\big|F_n(v) - F_n(u)\big| \leq L(x) \cdot |v - u|.$$

Proof: $(i) \Rightarrow (ii)$: There exists $L > 0$ such that for each $u, v \in [a, b]$ and for all $n \in \mathbb{N}$, we have

$$\big|F_n(v) - F_n(u)\big| \leq L \cdot |v - u|.$$

For each $x \in [a, b]$, let $L(x) = L > 0$. Consider the following cases:

Case 1: Suppose that $x = a$ (or $x = b$). Choose $\delta(x) = \frac{1}{2}(b - a)$. Let $([u, v], x)$ be a McShane δ-fine interval-point pair with $[u, v] \subseteq [a, b]$. Then for all $n \in \mathbb{N}$,

$$\big|F_n(v) - F_n(u)\big| \leq L \cdot |v - u| = L(x) \cdot |v - u|.$$

Case 2: If $x \in (a, b)$, then choose $\delta(x) = \min\{\frac{1}{2}|b - x|, \frac{1}{2}|x - a|\}$. Let $([u, v], x)$ be a McShane δ-fine interval-point pair with $[u, v] \subseteq [a, b]$. Then for all $n \in \mathbb{N}$,

$$\big|F_n(v) - F_n(u)\big| \leq L \cdot |v - u| = L(x) \cdot |v - u|.$$

$(ii) \Rightarrow (i)$: By (ii), $\delta(x)$ is a gauge on $[a, b]$. By Cousin's Lemma, there exists a McShane δ-fine division $D = \{([s_i, t_i], x_i)\}_{i=1}^{r}$ of $[a, b]$. Let $L = \max\{L(x_i) : i = 1, 2, \cdots, r\} > 0$. For each $i = 1, 2, \ldots, r$ and for all $n \in \mathbb{N}$,

$$\big|F_n(t_i) - F_n(s_i)\big| \leq L(x_i) \cdot |t_i - s_i| \leq L \cdot |t_i - s_i|.$$

Let $x, y \in [a, b]$ with $x < y$. Then there exists $1 \leq j \leq k \leq r$ such that $x \in [s_j, t_j]$ and $y \in [s_k, t_k]$. Let P be the McShane δ-fine partial division

$$\Big\{([x, t_j], x_j), ([s_{j+1}, t_{j+1}], x_{j+1}), \ldots, ([s_{k-1}, t_{k-1}], x_{k-1}), ([s_k, y], x_k)\Big\}.$$

Relabel P by $P = \{([s_\alpha, t_\alpha], \xi_\alpha)\}_{\alpha=1}^{q}$. Then $[x, y] = \displaystyle\bigcup_{\alpha=1}^{q} [s_\alpha, t_\alpha]$. Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned} \big|F_n(y) - F_n(x)\big| &= \left|\sum_{\alpha=1}^{q} \Big[F_n(t_\alpha) - F_n(s_\alpha)\Big]\right| \leq \sum_{\alpha=1}^{q} \Big|F_n(t_\alpha) - F_n(s_\alpha)\Big| \\ &\leq \sum_{\alpha=1}^{q} \Big(L \cdot |t_\alpha - s_\alpha|\Big) = L \cdot \sum_{\alpha=1}^{q} |t_\alpha - s_\alpha| = L \cdot |y - x|. \end{aligned}$$

This shows that F is uniformly Lipschitz on $[a, b]$. ■

Definition 3.3 Let $\varnothing \neq X \subseteq [a,b]$. A sequence $\langle F_n\rangle_{n=1}^{\infty}$ of functions on $[a,b]$ is said to satisfy a *uniformly δ-Lipschitz condition on* X if there exist $L>0$ and $\delta(x)>0$ on X such that for any δ-fine interval-point pair $([u,v],x)$ with $x \in X$, we have

$$|F_n(v) - F_n(u)| \leq L \cdot |v-u|, \quad \text{for all } n \in \mathbb{N}.$$

Theorem 3.4 *Let $\langle F_n\rangle_{n=1}^{\infty}$ be a sequence of functions on $[a,b]$. If there exists a sequence $\langle X_m\rangle_{m=1}^{\infty}$ of non-empty subsets of $[a,b]$ such that $[a,b] = \bigcup_{m=1}^{\infty} X_m$ and $\langle F_n\rangle_{n=1}^{\infty}$ is uniformly δ-Lipschitz on each X_m, then $\langle F_n\rangle_{n=1}^{\infty}$ is uniformly Lipschitz on $[a,b]$.*

Proof: For all $x \in [a,b]$, there exists $m_0 \in \mathbb{N}$ such that $x \in X_{m_0}$. Since $\langle F_n\rangle_{n=1}^{\infty}$ is uniformly δ-Lipschitz on each X_{m_0}, there exist $L_{m_0} > 0$ and $\delta_{m_0}(x) > 0$ on X_{m_0} such that for any δ_{m_0}-fine interval-point pair $([u,v],x)$ with $x \in X_{m_0}$, we have

$$|F_n(v) - F_n(u)| \leq L_{m_0} \cdot |v-u|, \quad \text{for all } n \in \mathbb{N}.$$

Choose $L(x) = L_{m_0}$ and $\delta(x) = \delta_{m_0}(x)$, for all $x \in X_{m_0}$. Let $([u,v],x)$ be δ-fine. Then

$$|F_n(v) - F_n(u)| \leq L_{m_0} \cdot |v-u| = L(x) \cdot |v-u|, \text{ for all } n \in \mathbb{N}.$$

Thus, by Theorem 3.2, $\langle F_n\rangle_{n=1}^{\infty}$ is uniformly Lipschitz on $[a,b]$. ■

Definition 3.5 [5] Let $X \subseteq [a,b]$. A sequence $\langle F_n\rangle_{n=1}^{\infty}$ of functions defined on $[a,b]$ is said to be $UAC^*(X)$ if for every $\epsilon > 0$ there exists $\eta > 0$, independent of n, such that for any partial partition $P = \{[a_k, b_k]\}$ of $[a,b]$ with $a_k, b_k \in X$,

$$(P)\sum |b_k - a_k| < \eta \quad \text{implies} \quad (P)\sum \omega(F_n;[a_k,b_k]) < \epsilon, \quad \text{for all } n,$$

where $\omega(F_n;[a_k,b_k])$ is the oscillation of F_n on $[a_k,b_k]$. The sequence $\langle F_n\rangle_{n=1}^{\infty}$ is $UACG^*$ *on* $[a,b]$ if $[a,b]$ is a union of X_k, $k=1,2,\ldots$, such that $\langle F_n\rangle_{n=1}^{\infty}$ is $UAC^*(X_k)$ for each k.

Theorem 3.6 *Let $\langle F_n\rangle_{n=1}^{\infty}$ be a sequence of functions on $[a,b]$. If there exists a sequence $\langle X_m\rangle_{m=1}^{\infty}$ of non-empty subsets of $[a,b]$ such that $[a,b] = \bigcup_{m=1}^{\infty} X_m$ and $\langle F_n\rangle_{n=1}^{\infty}$ is uniformly δ-Lipschitz on each X_m, then $\langle F_n\rangle_{n=1}^{\infty}$ is $UACG^*$ on $[a,b]$.*

Proof: Choose any X_m and fix this (but arbitrary). By hypothesis, there exist $L>0$ and $\delta(x)>0$ on X_m such that for any Henstock δ-fine interval-point pair $([u,v],x)$ with $x \in X_m$, we have

$$|F_n(v) - F_n(u)| \leq L \cdot |v-u|, \quad \text{for all } n.$$

For each $i,j \in N$, let

$$X_{m,i,j} = \left\{x \in X_m : \tfrac{1}{i+1} < \delta(x) \leq \tfrac{1}{i},\ x \in [a + \tfrac{j-1}{i+1}, a + \tfrac{j}{i+1})\right\}.$$

Then $\{X_{m,i,j} : i,j \in \mathbb{N}\}$ is pairwise-disjoint and $X_m = \bigcup_{i,j} X_{m,i,j}$.

Now, fix $X_{m,i,j}$. We will show that $\langle F_n \rangle_{n=1}^{\infty}$ is $UAC^*(X_{m,i,j})$. So let $\epsilon > 0$. Take $\eta = \dfrac{\epsilon}{L} > 0$. Let $P = \{[a_k, b_k]\}$ be a partial partition of $[a,b]$ with $a_k, b_k \in X_{m,i,j}$ and

$$\sum_k |b_k - a_k| < \eta.$$

Then, for every k,

$$|b_k - a_k| \leq \left(a + \tfrac{j}{i+1}\right) - \left(a + \tfrac{j-1}{i+1}\right) = \tfrac{1}{i+1} < \delta(a_k).$$

This implies that $([a_k, b_k], a_k)$ is δ-fine for all k. Thus, for each k

$$|F_n(b_k) - F_n(a_k)| \leq L \cdot |b_k - a_k|, \quad \text{for all } n.$$

Moreover, if $a_k \leq u_k \leq v_k \leq b_k$, then $([a_k, u_k], a_k)$ and $([v_k, b_k], b_k)$ are δ-fine. So, for all n

$$|F_n(u_k) - F_n(a_k)| \leq L \cdot |u_k - a_k| \leq L \cdot |b_k - a_k|$$

and

$$|F_n(b_k) - F_n(v_k)| \leq L \cdot |b_k - v_k| \leq L \cdot |b_k - a_k|.$$

Hence, for all n

$$\begin{aligned} |F_n(v_k) - F_n(u_k)| &\leq |F_n(v_k) - F_n(b_k)| + |F_n(b_k) - F_n(a_k)| + |F_n(a_k) - F_n(u_k)| \\ &\leq 3L \cdot |b_k - a_k|. \end{aligned}$$

For each k, it follows that

$$\omega(F_n; [a_k, b_k]) = \sup\left\{|F_n(v_k) - F_n(u_k)| : \ u_k, v_k \in [a_k, b_k]\right\} \leq 3L \cdot |b_k - a_k|,$$

for all n. Thus, for all $n \in \mathbb{N}$,

$$\sum_k \omega(F_n; [a_k, b_k]) \leq 3L \sum_k |b_k - a_k| < 3L \cdot \eta = \epsilon.$$

This shows that $\langle F_n \rangle_{n=1}^{\infty}$ is $UAC^*(X_{m,i,j})$ for all $i, j \in \mathbb{N}$. Since $[a,b]$ is countable union of $X_{m,i,j}$, it follows that $\langle F_n \rangle_{n=1}^{\infty}$ is $UACG^*$ on $[a,b]$. ■

4 Controlled Convergence Theorem

In this section, we utilize the concept of uniform Lipschitz to investigate the Controlled Convergence Theorem. The following result is known as the Controlled Convergence Theorem.

Theorem 4.1 (Controlled Convergence Theorem) [8] *Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of Henstock integrable functions on $[a,b]$ with corresponding primitives F_n and $\langle f_n \rangle_{n=1}^{\infty}$ converges to f pointwisely on $[a,b]$. If $\langle F_n \rangle_{n=1}^{\infty}$ is $UACG^*$, then f is Henstock integrable on $[a,b]$ and*

$$\lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_a^b f.$$

We present the following "stronger" version of the Controlled Convergence Theorem.

Theorem 4.2 (Controlled Convergence Theorem Version I) *Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of Henstock integrable functions on $[a,b]$ with corresponding primitives F_n and $\langle f_n \rangle_{n=1}^{\infty}$ converges to f pointwisely on $[a,b]$. Suppose there exists a sequence $\langle X_m \rangle_{m=1}^{\infty}$ of non-empty subsets of $[a,b]$ such that the following conditions are satisfied:*

(a) $[a,b] = \bigcup_{m=1}^{\infty} X_m$ *and*

(b) $\langle F_n \rangle_{n=1}^{\infty}$ *is uniformly δ-Lipschitz on each X_m.*

Then f is Henstock integrable and

$$\lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_a^b f.$$

Proof: By Theorem 3.6, $\langle F_n \rangle_{n=1}^{\infty}$ is $UACG^*$ on $[a,b]$. The conclusion follows from the Controlled Convergence Theorem. ∎

Theorem 4.3 (Controlled Convergence Theorem Version II) *Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of Henstock integrable functions on $[a,b]$ with corresponding primitives F_n and $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwisely to f on $[a,b]$. Suppose there exists subsets X_0 and X_m $(m \geq 1)$ of $[a,b]$ such that the following conditions are satisfied:*

(a) $[a,b] \setminus X_0 = \bigcup_{m=1}^{\infty} X_m$,

(b) $\langle F_n \rangle_{n=1}^{\infty}$ *is $UAC^*(X_0)$, and*

(c) $\langle F_n \rangle_{n=1}^{\infty}$ *is uniformly δ-Lipschitz on each X_m, for $m \geq 1$.*

Then f is Henstock integrable and

$$\lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_a^b f.$$

Proof: Since $\langle F_n \rangle_{n=1}^{\infty}$ is $UAC^*(X_0)$ and

$$[a,b] = \bigcup_{m=0}^{\infty} X_m,$$

it follows from Theorem 3.6 that $\langle F_n \rangle_{n=1}^{\infty}$ is $UACG^*$ on $[a,b]$. Therefore, by Controlled Convergence Theorem, f is Henstock integrable and

$$\lim_{n\to\infty} (\mathcal{H}) \int_a^b f_n = (\mathcal{H}) \int_a^b f. \qquad ∎$$

Acknowledgments: The authors are grateful to the anonymous referees for their helpful comments and suggestions.

References

[1] R.G. Bartle and D.R. Sherbert, *Introduction to Real Analysis*, John Wiley and Sons, Inc., (2011).

[2] J. Kurzweil and J. Jarnìk, *Equi-integrability and controlled convergence of Perron-type integrable functions*, Real Analysis Exchange 17, 110-139, (1992).

[3] Z. Gong and Y. Shao, *The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions*, Fuzzy Sets and Systems 160(11), 1528-1546, (2009).

[4] E. Mema, *Equiintegrability and Controlled Convergence for the Henstock-Kurzweil Integral*, International Mathematical Forum 8(19), 913-919, (2013).

[5] P.Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, 1989.

[6] P.Y. Lee and T. S. Chew, *A better convergence theorem for Henstock integrals*, Bull. London Math. Soc. 17, 557-574, (1985).

[7] P.Y. Lee and T.S. Chew, *A short proof of the controlled convergence theorem for Henstock integrals*, Bull. London Math. Soc. 19, 60-62, (1987).

[8] P.Y. Lee, and R. Yyborny, *The integral: an easy approach after Kurzweil and Henstock*, Cambridge University Press, (2000).

[9] S. Nakanishi, *On controlled convergence theorem for Henstock-Kurzweil integrals*, Southeast Asian Bulletin of Math. 14, 135-141, (1990).

[10] S. Pal, D.K. Ganguly and L.P. Yee, *A generalized Henstock-Stieltjes Integral Involving Division Functions*, Mathematica Slovaca 58(4), 413-438, (2008).

[11] L.I. Paredes and T.S. Chew, *Controlled convergence theorem for Banach-valued HL-integrals*, Scientiae Mathematicae Japonicae 56, 347-357, (2002).

[12] L.I. Paredes, P.Y. Lee and T.S. Chew, *Controlled convergence theorem for strong variational Banach-valued multiple integrals*, Real Analysis Exchange 28(2), 579-592, (2002-03).

[13] Y. Shao and H. Zhang, *The Strong Fuzzy Henstock Integrals and Discontinuous Fuzzy Differential Equations*, Journal of Applied Mathematics 2013, 1-8, (2013).

[14] P. Wang, *Equi-integrability and Controlled convergence for the Henstock integral*, Real Analysis Exchange, 19(1), 236-241, (1993-94).