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On the I -Integral of Graphs Under Some Binary Operations

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Abstract

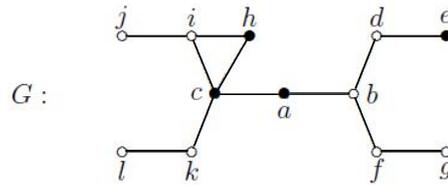
Let $G = (V(G), E(G))$ be an undirected connected graph and let X be a subset of $V(G)$. Furthermore, let $I(X)$ and $B(X)$ denote the set of isolates and the boundary set of X , respectively. The *inner boundary number* of X , denoted by $\beta_i(X)$ is $\beta_i(X) = \max\{|Y| : Y \subseteq X \text{ and } B(X \setminus Y) \setminus Y = B(X)\}$. The *outer boundary number* of X , denoted by $\beta_o(X)$ is $\beta_o(X) = |V(G) \setminus N[X]|$. The I -integral of X is $\int_I(X) = \beta_i(X) + \beta_o(X) + |I(X)|$ and the I -integral of G is $\int_I(G) = \min\{\int_I(X) : X \subseteq V(G)\}$. In this paper, we determine the I -integral of graphs resulting from some binary operations such as the join, corona, composition, and cartesian product of graphs.

Key words: I -integral, isolates, boundary

Mathematics Subject Classification: 05C69

1 Introduction

Consider a network of locations. We may represent it as a graph. Suppose we want to build production facilities in some of these locations. Each of these production facilities can stand on its own but it has better production if connected to another facility. These production facilities can distribute their products only to locations directly connected to it. We further assume that the gain we may have in having one location for distribution is equal to the gain of having a production facility connected to another facility or the gain of having one less “unneded” production facility. Given a set of locations for your production facilities you may count the locations that are not connected to any production facility, the number of “unneded” facilities, and the number of isolated facilities. What we want is the minimum of the sum of these numbers. For example, consider the network of locations represented by a graph G below.



If you put four production facilities at locations $a, c, h,$ and e , the locations that are not connected to any production facility are $j, l, f,$ and g . The production facility located at e is an isolated facility because it is not connected to any other facility. The production facility located at h is an “unnneeded” facility because its set of locations for product distribution, which is just location i , is already covered by the production facility in c . Thus, the number of locations that are not connected to any production facility is 4 , the number of isolated facilities is 1 , and the number of “unnneeded” facilities is 1 . Summing up these numbers we have 6 , and we associate this number as the I -integral of the set of locations $X = \{a, c, h, e\}$. We may consider all other set of locations, and the minimum of the I -integral of all these set of locations is what we mean as the I -integral of G . These concepts were already defined precisely in³. For the sake of completeness, we will give it again in the next section.

2 Preliminaries :

Let G be an undirected connected graph and let X be a subset of $V(G)$. We first recall the following definitions.

The set $I(X) = X \setminus N(X)$ denotes the set of isolates in X and $B(X) = N(X) \setminus X$ denotes the boundary of X , where $N(X) = \{y \in V(G) : yx \in E(G) \text{ for some } x \in X\}$ is the set of neighbors of X . Let $N[X] = N(X) \cup X$. A set $S \subseteq V(G)$ is a *dominating set* of G if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. If S is a dominating set with $|S| = \gamma(G)$, then we call S a *minimum dominating set* of G or a γ -set in G . If $N(S) = V(G)$, then we say that S is a *total dominating set* of G . The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. If S is a total dominating set with $|S| = \gamma_t(G)$, then we call S a *minimum total dominating set* of G or a γ_t -set in G .

Definition 2.1 The *inner boundary number* of X , denoted by $\beta_i(X)$ is

$$\beta_i(X) = \max\{|Y| : Y \subseteq X \text{ and } B(X \setminus Y) \setminus Y = B(X)\}.$$

The *outer boundary number* of X , denoted by $\beta_o(X)$ is

$$\beta_o(X) = |V(G) \setminus N[X]|.$$

The I -integral of X is $\int_I(X) = \beta_i(X) + \beta_o(X) + |I(X)|$ and the I -integral of G is

$$\int_I(G) = \min \left\{ \int_I(X) : X \subseteq V(G) \right\}.$$

We will call $Y \subseteq X$ such that $\beta_i(X) = |Y|$ an *inner boundary set* of X and the set $V(G) \setminus N[X]$ the *outer*

boundary set of X .

Remark 2.2 : It is easy to show that the equation $B(X \setminus Y) \setminus Y = B(X)$ is equivalent to $N(X \setminus Y) \setminus X = N(X) \setminus X$.

Observe that for a connected graph G of order $n \geq 2$, if we take $X = \emptyset$, $\beta_o(X) = |V(G)| = n$ and $\beta_i(X) = |I(X)| = 0$. Thus, $\int_r(X) = n$. If $X = V(G)$, then $\beta_o(X) = |I(X)| = 0$. Note that $B(X) = N(X) \setminus X = V(G) \setminus V(G) = \emptyset$ and $B(X \setminus X) \setminus X = \emptyset$. Thus, $\beta_i(X) = |V(G)| = n$. Thus, $\int_r(X) = n$.

Let $\emptyset \subset X \subset V(G)$ and $Y \subseteq X$. If $Y = \emptyset$, then $B(X \setminus Y) \setminus Y = B(X)$. If $Y = X$, then, since $X \neq V(G)$, there exists a $y \in V(G)$ not in X and since G is connected, there exists a path connecting y to X . The vertex in this path not in X but directly connected to X is an element of $N(X)$. Hence, $N(X) \setminus X = B(X) \neq \emptyset$. But $B(X \setminus X) \setminus X = \emptyset$. Thus, $0 \leq \beta_i(X) \leq |X| - 1$.

We state it as a remark.

Remark 2.3 For a connected graph G of order $n \geq 2$ and $X = \emptyset$ or $X = V(G)$, $\int_r(X) = n$. Moreover, if $\emptyset \subset X \subset V(G)$, then $0 \leq \beta_i(X) \leq |X| - 1$.

For the graph G of order 1, say $V(G) = \{a\}$, the set $X \subseteq V(G)$ is either \emptyset or $\{a\}$. If $X = \emptyset$, then $\beta_i(X) = |I(X)| = 0$ and $\beta_o(X) = 1$. If $X = \{a\}$, then $|I(X)| = 1$ and $\beta_o(X) = 0$. Moreover, you may take $Y = \{a\}$ and $B(X \setminus Y) \setminus Y = \emptyset = B(X)$. Thus, $\beta_i(\{a\}) = 1$ and $\int_r(\{a\}) = 2$. Hence, $\int_r(G) = \int_r(\emptyset) = 1$.

For the connected graph G of order 2, say $V(G) = \{a, b\}$, the set $X \subseteq V(G)$ is either \emptyset , $\{a\}$, $\{b\}$, or $V(G)$. By Remark 2.3, $\int_r(\emptyset) = \int_r(V(G)) = 2$. Let $X = \{a\}$ or $\{b\}$. Then, $\beta_o(X) = 0$ and $|I(X)| = 1$. Moreover, by the second part of Remark 2.3, $\beta_i(X) = 0$. Hence, $\int_r(X) = 1$ and $\int_r(G) = 1$. We thus have the following remark.

Remark 2.4 : For the graph G of order 1 or 2, $\int_r(G) = 1$.

The following results are from ³. The first theorem shows the bounds for the I -integral of a graph and the second asserts the existence of a graph with I -integral equal to a natural number n . To see the construction of such a graph, we reproduce the proof here.

We first recall that the *degree of a vertex* v of a graph G is given by $deg(v) = |N(\{v\})|$ and the *degree of* G , denoted by $\Delta(G)$, is $\max\{deg(v) : v \in V(G)\}$.

*Theorem 2.5*³ For any connected graph G of order $n \geq 3$,

$$0 \leq \int_r(G) \leq n - \Delta(G).$$

*Theorem 2.6*³ For any natural number n , there exists a connected graph G with $\int_r(G) = n$.

Proof. Let $n \geq 1$. Start with the path $P_3 = [v_1^1, v_2^1, v_3^1]$. For vertices v_1^1 and v_3^1 connect additional n vertices.

Let this new graph be G . Make n copies of G , labeling the vertices corresponding to v_1^1, v_2^1, v_3^1 as $v_1^2, v_2^2, v_3^2, v_1^3, v_2^3, v_3^3, \dots, v_1^n, v_2^n, v_3^n$. For $k = 1, 2, \dots, n-1$, connect one additional vertex connected to v_3^k to one additional vertex connected to v_1^{k+1} . Let this new connected graph be G_n . We are to show that $\int_r(G_n) = n$.

Let X_1 be the vertices $\{v_1^1, v_3^1, v_1^2, v_3^2, \dots, v_1^n, v_3^n\}$ and X_2 be the vertices $\{v_2^1, v_2^2, v_2^3, \dots, v_2^n\}$. Let $X = X_1 \cup X_2$. Then, X is a total dominating set and thus we have $\beta_o(X) = 0$ and $I(X) = \emptyset$. Now, consider that $X_2 \subseteq X$ and $B(X \setminus X_2) \setminus X_2 = B(X)$. It can be easily seen that X_2 is the largest subset of X having that property, and hence, $\beta_i(X) = n$. Hence, $\int_r(X) = \beta_o(X) + \beta_i(X) + |I(X)| = 0 + n + 0 = n$.

Let $Y \subseteq V(G_n)$ such that $Y \neq X$. If $X \subset Y$, then Y is also a total dominating set and hence $\beta_o(Y) = 0$ and $I(Y) = \emptyset$ and $\beta_i(Y) \geq n$. Hence, $\int_r(Y) \geq n$. If X is not a subset of Y , then at least a vertex in X_1 or in X_2 is not in Y . If a vertex in X_1 is not in Y , then $\beta_o(Y) \geq n$. If $X_1 \subset Y$ and $v_2^k \in X_2$ is not in Y , then v_1^k and v_3^k which are vertices in X_1 are isolates. If we let S be the rest of X_2 in Y , we have $\beta_i(Y) = |S|$. Hence, $\beta_i(Y) + |I(Y)| \geq |X_2| = n$. Thus, $\int_r(Y) \geq n$. Therefore, $\int_r(G) = n$, by the minimality of $\int_r(G)$. \square

The following are rectifications of some of the results from ³. The next theorem and its corollary rectify Theorem 2.2 and Corollary 2.3 in ³.

Theorem 2.7: Let G be a connected graph of order $n \geq 3$.

Then, $\int_r(G) \leq \gamma_t(G) - 1$.

Proof. Let $X \subseteq V(G)$ such that $N(X) = V(G)$ and $\gamma_t(G) = |X|$. Then, $\beta_o(X) = |V(G) \setminus N[X]| = 0$ and $I(X) = \emptyset$. Let $Y \subseteq X$ such that $B(X \setminus Y) \setminus Y = B(X)$. Observe that Y cannot be equal to X , since $B(X \setminus X) = \emptyset$. Hence, $0 \leq |Y| \leq |X| - 1 = \gamma_t(G) - 1$. Thus,

$$\begin{aligned} \int_r(G) &\leq \int_r(X) = \beta_o(X) + \beta_i(X) + |I(X)| \\ &= 0 + |Y| + 0 \leq \gamma_t(G) - 1. \end{aligned} \quad \square$$

Corollary 2.8 Let G be a connected graph of order $n \geq 3$.

Then, $\int_r(G) + \partial_r(G) \leq n - 1$.

The notation $\partial_r(G)$ is for the I -differential of a graph G , defined as $\partial_r(G) = \max\{\partial_r(X) : X \subseteq V(G)\}$, where $\partial_r(X) = |B(X)| - |I(X)|$. The author had worked on this parameter with Canoy in ⁴. This work had inspired the author to develop the I -integral of a graph.

The next theorem rectifies Theorem 2.5 in ³.

Theorem 2.9 Let G be a connected graph of order $n \geq 3$. Then, $\int_r(G) = 0$ if and only if G has a total

dominating set S with the property that for all $x \in S$, $N(\{x\}) \setminus S \not\subseteq N(S \setminus \{x\}) \setminus S$.

Proof. Suppose that $\int_I(G) = 0$. Then, there exists $S \subseteq V(G)$ such that $\beta_o(S) = \beta_i(S) = 0$ and $I(S) = \emptyset$. Since $\beta_o(S) = 0$, we have $N[S] = V(G)$. Moreover, since S has no isolates, $S \subseteq N(S)$ and thus, $N[S] = N(S) \cup S = N(S)$, that is, $N(S) = V(G)$. Thus, S is a total dominating set.

Now, suppose that there exists a vertex $x \in S$ such that $N(\{x\}) \setminus S \subseteq N(S \setminus \{x\}) \setminus S$. That is, all neighbors of x not in S are also connected to the other members of S . Thus, $N(S) \setminus S = B(S) = N(S \setminus \{x\}) \setminus S = B(S \setminus \{x\}) \setminus \{x\}$. Hence, $\beta_i(S) \geq 1$, a contradiction. Therefore, for all $x \in S$, $N(\{x\}) \setminus S \not\subseteq N(S \setminus \{x\}) \setminus S$.

Conversely, if S is a total dominating set of G , then $S \neq \emptyset$ and $\beta_o(S) = |I(S)| = 0$. Since for all $x \in S$, $N(\{x\}) \setminus S \not\subseteq N(S \setminus \{x\}) \setminus S$, then $S \neq V(G)$ and for every $x \in S$ there is at least a neighbor of x not in S and not connected to the other members of S . Hence, $N(S \setminus \{x\}) \setminus S = B(S \setminus \{x\}) \setminus \{x\} \neq N(S) \setminus S = B(S)$. Thus, for $\emptyset \subset Y \subset S$, $B(S \setminus Y) \setminus Y \neq B(S)$. Hence, $\beta_i(S) = 0$. Consequently, $\int_I(G) = 0$. \square

3 Results

Remark 3.1 : As discussed in the introduction and in view of Theorem 2.5, a desirable network of locations (a graph G), based on our problem, is a graph G with $\int_I(G) = 0$. We will see in this section that it is the case for some graphs under binary operations.

We will first obtain the I -integral of a complete graph.

Proposition 3.2 For the complete graph K_n , $\int_I(K_n) = 1$.

Proof. By Remark 2.4, $\int_I(K_n) = 1$, for $n = 1, 2$. Let $n \geq 3$ and $v \in V(K_n)$. Take $X = \{v\}$. Then, $\beta_o(X) = |V(G) \setminus N[X]| = |V(G) \setminus V(G)| = 0$ and $|I(X)| = 1$. Moreover, by the second part of Remark 2.3, $\beta_i(X) = 0$. Hence, $\int_I(X) = 1$. If $X = \emptyset$ or $V(K_n)$, then by Remark 2.3, $\int_I(X) = n > 1$.

Let $X \subset V(G)$, $|X| \geq 2$ and $v \in X$. If we let $Y = X \setminus \{v\}$, then, since all the other vertices are connected to v , we have $N(X \setminus Y) \setminus X = N(\{v\}) \setminus X = N(X) \setminus X$. Thus, in view of Remark 2.2 and the second part of Remark 2.3, $\beta_i(X) = |Y| \geq 1$. Hence, $\int_I(X) \geq 1$. Therefore, $\int_I(G) = 1$. \square

Let us recall the definitions for the join and corona of graphs.

Let A and B be sets which are not necessarily disjoint. The *disjoint union* of A and B , denoted by $A \dot{\cup} B$, is the set obtained by taking the union of A and B treating each element in A as distinct from each element in B . The *join* of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Let G and H be graphs of order n and m , respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and n copies of H , and then joining the i th vertex of G to every vertex of the i th copy of H . For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are attached one by one to the vertex v .

For connected graphs G and H of orders $1 \leq n, m \leq 2$, respectively, the resulting graph for $G + H$ is a complete graph, which is already solved by the preceding proposition and thus we will exclude it in the next theorem.

Theorem 3.3: Let G and H be connected graphs of orders $n \geq 1$, $m \geq 3$, respectively. Then,

$$\int_I(G + H) = \begin{cases} 0, & \text{if both } G \text{ and } H \text{ are not complete graphs or,} \\ & \text{if } \int_I(G) = 0 \text{ or } \int_I(H) = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Proof: For the first case, it is enough to find an $X \subseteq V(G + H)$ such that $\int_r(X) = 0$. Suppose both G and H are not complete graphs. Then, there exist $v \in V(G)$ and $w \in V(H)$ such that $N[\{v\}] \neq V(G)$ and $N[\{w\}] \neq V(H)$. Take $X = \{v, w\}$. Note that $vw \in E(G + H)$. Hence, $I(X) = \emptyset$. Moreover, since all of $V(G)$ is connected to w and all of $V(H)$ is connected to v , then $N[X] = V(G + H)$, and thus $\beta_o(X) = 0$. For the computation of $\beta_i(X)$, let $Y \subseteq X$. In view of the second part of Remark 2.3, it is enough to consider $Y = \{v\}$ or $Y = \{w\}$. Without loss of generality, let $Y = \{v\}$. Then, $X \setminus Y = \{w\}$. All of $V(G)$ is connected to $\{w\}$ but there exists a $w_0 \in V(H)$ that is not connected to $\{w\}$, since $N[\{w\}] \neq V(H)$. Hence, $w_0 \notin N(X \setminus Y) \setminus X$. However, since all of $V(H)$ is connected to $\{v\}$, $w_0 \in N(\{v, w\})$ and thus, $w_0 \in N(X) \setminus X = V(G + H) \setminus \{v, w\}$. Therefore, $N(X \setminus Y) \setminus X \neq N(X) \setminus X$ and thus with Remark 2.2, we can conclude that $\beta_i(X) = 0$. Hence, $\int_r(X) = 0$.

Now, suppose that $\int_r(G) = 0$ or $\int_r(H) = 0$. Without loss of generality, let $\int_r(G) = 0$ and $X \subseteq V(G)$ such that $\int_r(X) = 0$. Then, $\beta_o(X) = 0$, $\beta_i(X) = 0$, and $I(X) = \emptyset$ with respect to G . Moreover, $X \subseteq V(G + H)$ and all of $V(H)$ is connected to X , hence $N[X] = V(G + H)$ and $\beta_o(X) = |V(G + H) \setminus N[X]| = 0$. Clearly, we also have $I(X) = \emptyset$ with respect to $G + H$. To solve for $\beta_i(X)$ with respect to $G + H$, let $Y \subseteq X$. Again, in view of the second part of Remark 2.3, it is enough to consider $\emptyset \neq Y \subset X$. Let $X_0 = X \setminus Y \neq \emptyset$. Since $\beta_i(X) = 0$ with respect to G , we have $N(X \setminus Y) \setminus X \neq N(X) \setminus X$ with respect to G , in view of Remark 2.2. Thus, there exists a $v \in V(G) \setminus X$ such that $v \in N(X)$ but $v \notin N(X \setminus Y)$. Since $X \cap V(H) = \emptyset$, we also have $v \notin N(X \setminus Y)$ with respect to $G + H$. But $v \in N(X)$, with respect to

$G + H$. Thus, with respect to $G + H$, we also have $N(X \setminus Y) \setminus X \neq N(X) \setminus X$. Therefore, $\beta_i(X) = 0$ with respect to $G + H$. Thus, $\int_i(X) = 0$.

For the second case, suppose that G or H is a complete graph and, $\int_i(G) \neq 0$ and $\int_i(H) \neq 0$. Without loss of generality, suppose that G is the complete graph. Let $v \in V(G)$ and $X_0 = \{v\}$. Then, in $G + H$, all of $V(H)$ is connected to v and since G is complete, we should have $N[\{v\}] = V(G + H)$ and thus $\beta_o(X_0) = 0$. Moreover, by the second part of Remark 2.3, $\beta_i(X_0) = 0$. Since $I(X_0) = \{v\}$, we have $\int_i(X_0) = 1$. Now, let $X \subseteq V(G + H)$. We are to show that $\int_i(X) \geq 1$. If $X = \emptyset$ and $X = V(G + H)$, then $\int_i(X) = n + m$, by Remark 2.3. Suppose $\emptyset \neq X \subset V(G + H)$. If $|X| = 1$, then $|I(X)| = 1$ and hence $\int_i(X) \geq 1$. Suppose $|X| \geq 2$. Then, either $X \subseteq V(H)$ or $X \not\subseteq V(H)$. If $X \not\subseteq V(H)$, then there exists a $u \in X$ such that $u \in V(G)$. Note that all of $V(H)$ is connected to u , and since G is complete, we have $N[\{u\}] = V(G + H)$. We then want to solve $\beta_i(X)$. Let $Y = X \setminus \{u\}$. Since X contains vertices from both G and H , we have $N[X] = V(G + H)$. Thus, $N(X) \setminus X = V(G + H) \setminus X = N[\{u\}] \setminus X = N(X \setminus Y) \setminus X$. Thus, in view of Remark 2.2 and the second part of Remark 2.3, $\beta_i(X) = |X| - 1 \geq 1$ and thus $\int_i(X) \geq 1$. We are left to consider the case $X \subseteq V(H)$. Since, $\int_i(H) \neq 0$, $\int_i(X) \geq 1$, with respect to H . Then, at least one of $\beta_o(X)$, $\beta_i(X)$, and $|I(X)|$ is greater than or equal to 1, with respect to X . Since X has no element from $V(G)$, an isolate of X with respect to H is also an isolate of X with respect to $G + H$. All of $V(G)$ is connected to X in $G + H$, and hence we are only left to consider $V(H)$ to determine $\beta_o(X)$ for $G + H$. Thus, $\beta_o(X)$ is the same for both H and $G + H$. In $G + H$, since $V(G)$ is connected to any vertex in H , the set $N(X)$ will only differ from $N(X \setminus Y)$ in $V(H)$, for $(X \setminus Y) \neq \emptyset$. Hence, $\beta_i(X)$ is also equal for both H and $G + H$. Thus, the $\int_i(X)$ is equal for both H and $G + H$. That is, the $\int_i(X) \geq 1$, with respect to $G + H$, and the proof is complete. \square

Observe that for connected graphs G and H of orders $n = 1$ and $m = 1, 2$, the corona $G \circ H$ is a complete graph, and thus its I -integral is equal to 1, by Proposition 3.2. For $n = 1$ and $m \geq 3$, $G \circ H$ is the same as $G + H$, and thus its I -integral can be determined by Theorem 3.3. We will exclude those cases in the following theorem.

Theorem 3.4 Let G and H be connected graphs of orders $n \geq 2$, $m \geq 1$, respectively. Then, $\int_i(G \circ H) = 0$.

Proof. Just like in the first case of the preceding theorem, it is enough to find an $X \subseteq V(G \circ H)$ such that $\int_i(X) = 0$. Take $X = V(G)$. Since G is connected and $n \geq 2$, we have $I(X) = \emptyset$. Clearly, $N[X] = V(G \circ H)$. Hence, $\beta_o(X) = 0$. To determine $\beta_i(X)$, note that $N(X) \setminus X = \cup_{v \in V(G)} V(H^v)$. If

$\emptyset \neq Y \subset X$, then $N(X \setminus Y) \setminus X = \cup_{v \in V(G) \setminus Y} V(H^v)$. That is, $N(X) \setminus X \neq N(X \setminus Y) \setminus X$. Thus, in view of Remark 2.2 and the second part of Remark 2.3, $\beta_i(X) = 0$. Therefore, $\int_I(G \circ H) = \int_I(X) = 0$. \square

Remark 3.5 The set X we use in the preceding two theorems to show that the I -integral is equal to 0 is a total dominating set with the property stated in Theorem 2.9

Let us now consider the composition of two graphs. Recall that the *composition* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Theorem 3.6 Let G and H be connected graphs of orders $n, m \geq 2$, respectively. Then, $\int_I(G[H]) = 0$ if H is not a complete graph. If H is a complete graph, then $\int_I(G[H]) \leq \beta_i(S)$, where S is a minimum total dominating set of G .

Proof. Suppose H is not a complete graph. Then, there exists a $v \in V(H)$ such that $\deg(v) \neq m - 1$. Let S be a minimum total dominating set of G . Then, $I(S) = \emptyset$. We denote by S_v the set $\{(s, v) : s \in S\}$. Note that $S_v \subseteq V(G[H])$. By the definition of composition, the induced subgraph of S_v in $G[H]$ is isomorphic to the induced subgraph of S in G . Thus, $I(S_v) = \emptyset$. Moreover, since $N(S) = V(G)$, we also have $N(S_v) = V(G[H])$. Thus, $\beta_o(S_v) = 0$ and S_v is a total dominating set of $G[H]$. Now, let $(s, v) \in S_v$. Then, $s \in S$ and S being a minimum total dominating set implies that $N(S \setminus \{s\}) \neq V(G)$. Let $y \in V(G) \setminus N(S \setminus \{s\})$. Then, $y \in N(\{s\})$ and, either $y \in S \setminus \{s\}$ (as an isolate in $S \setminus \{s\}$) or $y \notin S \setminus \{s\}$ (or $y \notin S$, since $y \neq s$). This implies that in $G[H]$, $(y, v) \in N(\{(s, v)\})$ but $(y, v) \notin N(S_v \setminus \{(s, v)\})$. If $y \notin S$, then $(y, v) \notin S_v$. Thus, we will have $(y, v) \in N(\{(s, v)\}) \setminus S_v$ but $(y, v) \notin N(S_v \setminus \{(s, v)\}) \setminus S_v$. That is, $N(\{(s, v)\}) \setminus S_v \not\subseteq N(S_v \setminus \{(s, v)\}) \setminus S_v$. If $y \in S \setminus \{s\}$ (as an isolate), then $(y, v) \in S_v \setminus \{(s, v)\}$ (as an isolate). Since $y \in N(\{s\})$, by the definition of composition, $(y, v') \in N(\{(s, v)\})$ for all $v' \in V(H)$. Since $\deg(v) \neq m - 1$, there exists a $v'' \in V(H)$ such that v'' is not connected to v . Note that $(y, v'') \in N(\{(s, v)\})$. However, since y is an isolate and $vv'' \notin E(H)$, we have $(y, v'') \notin N(S_v \setminus \{(s, v)\})$. Thus, in this case, we also have $N(\{(s, v)\}) \setminus S_v \not\subseteq N(S_v \setminus \{(s, v)\}) \setminus S_v$. Therefore, by Theorem 2.9, $\int_I(G[H]) = 0$.

For the second case, where H is a complete graph, we proceed just like the first case but we will just take any $v \in V(H)$, since for all $v \in V(H)$, $\deg(v) = m - 1$. Then, following exactly the same argument as above, we will have $I(S_v) = \emptyset$ and $\beta_o(S_v) = 0$. It remains to show that $\beta_i(S_v) \leq \beta_i(S)$. Let Y be an inner boundary set of S . Then, $\beta_i(S) = |Y|$. Suppose there exists a $Z_v \subseteq S_v$, that is $Z \subseteq S$, such that $|Z| > |Y|$ and $N(S_v \setminus Z_v) \setminus S_v = N(S_v) \setminus S_v$. Then, if $(u, v) \in N(Z_v) \setminus S_v$, then $(u, v) \in N(S_v \setminus Z_v)$. Hence, if $u \in N(Z) \setminus S$,

then $u \in N(S \setminus Z)$. That is, $N(S \setminus Z) \setminus S = N(S) \setminus S$. This implies that $\beta_i(S) \geq |Z| > |Y|$. A contradiction to the maximality of Y , hence, $\beta_i(S_v) \leq |Y| = \beta_i(S)$. Therefore, $\int_i(G[H]) \leq \int_i(S_v) = \beta_i(S_v) \leq \beta_i(S)$. \square

The following corollary follows directly from the second case of Theorem 3.6.

Corollary 3.7 Let G and H be connected graphs of orders $n, m \geq 2$, respectively. If H is a complete graph, then $\int_i(G[H]) \leq \min\{\beta_i(S) : S \text{ is a minimum total dominating set of } G\}$.

We will now consider the cartesian product of two graphs. Recall that the cartesian product $G \times H$ of two graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H)$ satisfying the following condition: $(u, u')(v, v') \in E(G \times H)$ if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

Theorem 3.8 Let G and H be connected graphs of orders $n, m \geq 2$, respectively. Then, $\int_i(G \times H) \leq m\beta_i(S)$, where S is a minimum dominating set of G and $\int_i(G \times H) \leq n\beta_i(X)$, where X is a minimum dominating set of H .

Proof. Let S is be a minimum dominating set of G . Then, $N[S] = V(G)$. Just like in the preceding theorem, we will use the notation X_v for the set $\{(x, v) : x \in X\}$, if $X \subseteq V(G)$ and $v \in V(H)$ or for the set $\{(v, x) : x \in X\}$, if $X \subseteq V(H)$ and $v \in V(G)$. Let $K = \cup_{v \in V(H)} S_v$. Then, $K \subseteq V(G \times H)$ and $N[K] = V(G \times H)$. Hence, $\beta_o(K) = 0$. Since H is connected and of order $m \geq 2$, any $(s, v) \in K$ is connected to some $(s, v') \in K$. Hence, $I(K) = \emptyset$. Now, let Y_K be an inner boundary set of K . That, is $\beta_i(K) = |Y_K|$. Then, for every $v \in V(H)$, $|Y_K \cap S_v| \leq \beta_i(S)$. Hence, $|Y_K| \leq m\beta_i(S)$. Therefore, $\int_i(G \times H) \leq \int_i(K) = \beta_i(K) \leq m\beta_i(S)$.

Let X be a minimum dominating set of H . Take $K = \cup_{v \in V(G)} X_v$. Then, following the arguments above, but just exchanging the roles of G and H , we should have $\int_i(G \times H) \leq n\beta_i(X)$. \square

The following corollary follows directly from Theorem 3.8.

Corollary 3.9 Let G and H be connected graphs of orders $n, m \geq 2$, respectively. Then, $\int_i(G[H]) \leq \min(\{m\beta_i(S) : S \text{ is a } \gamma\text{-set of } G\} \cup \{n\beta_i(X) : X \text{ is a } \gamma\text{-set set of } H\})$.

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