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Simple Properties of PUL-Stieltjes Integral in Banach Space

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Abstract

Using PUL integrals, Boonpogkrong in [2] defined and discussed the Kurzweil-Henstock integral on manifolds. In this paper, we introduce the PUL-Stieltjes integral of Banach-valued functions and give some simple properties of this integral. Moreover, a characterization of PUL-Stieltjes integral is also given by establishing the Cauchy criterion.

1 Introduction

In [2], Kurzweil-Henstock integral on manifolds is defined using partition of unity, a concept introduced by J. Kurzweil and J. Jarnik in [3]. In the said paper, the authors defined the PUL integral and proved its equivalence to the Lebesgue integral in $\mathbb{R}^n$ using lower and upper semi-continuous functions.

In classical theory, integration on manifolds is done by *change of variables* and the PUL integral can be used in such process. Although the PUL integral of a real-valued function $f : M \to \mathbb{R}$ defined on *compact differentiable* r*-manifold* M is defined using an *atlas* Θ, its value is independent in the choice of Θ.

In this paper, we introduce and discuss some of the simple properties of the PUL-Stieltjes integral of functions taking values in a Banach space. Properties such as uniqueness, homogeneity, and linearity of both the integrands and integrators are proved. Moreover, Cauchy criterion for the PUL-Stieltjes integral is formulated and used to characterize such integrals.

2 PUL-Stieltjes Integral in Banach Space

In what follows, we denote a compact interval in $\mathbb{R}^n$ by $[\boldsymbol{a},\boldsymbol{b}] = \prod_{k=1}^{n}[a_k,b_k]$ with $[a_k,b_k] \subseteq \mathbb{R}$ for each $k = 1,2,\ldots,n$ and $\mu([\boldsymbol{a},\boldsymbol{b}]) = \prod_{k=1}^{n}(b_k - a_k)$ be the volume of $[\boldsymbol{a},\boldsymbol{b}]$. Moreover, $\mathbb{R}^n$ is equipped with the norm $\|\cdot\|_n$ defined by

$$\|\boldsymbol{x}\|_n = \max\{|x_i| : i = 1,2,\cdots,n\}$$

and for $r > 0$, we write $B(\boldsymbol{x};r) = \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{y}\|_n < r\}$, where

$$\boldsymbol{x} - \boldsymbol{y} = (x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n)$$

for $\boldsymbol{x} = (x_1,x_2,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,y_2,\ldots,y_n)$.

For a smooth function $\psi : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$, the *support* of ψ, denoted by supp ψ, is given by

$$\text{supp } \psi = \overline{\{\boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}] : \psi(\boldsymbol{x}) \neq 0\}},$$

where $\overline{A}$ denotes the closure of $A \subseteq \mathbb{R}^n$. A *gauge* on $[\boldsymbol{a},\boldsymbol{b}]$ is a positive function defined on $[\boldsymbol{a},\boldsymbol{b}]$.

Definition 2.1 [2] A finite collection $\{\psi_k\}_{k=1}^m$ of smooth functions defined on $[\boldsymbol{a},\boldsymbol{b}]$ is said to be a *partial partition of unity* if the following holds:

1. $\psi_k(\boldsymbol{\xi}) \geq 0$ for all $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$ and for all $k \in \{1,2,\cdots,m\}$ and
2. $\sum_{k=1}^{m}\psi_k(\boldsymbol{\xi}) \leq 1$ for all $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$.

If $\sum_{k=1}^{m}\psi_k(\boldsymbol{\xi}) = 1$ for all $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$, then $\{\psi_k\}_{k=1}^m$ is said to be a *partition of unity.*

Definition 2.2 [2] Let $\psi : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ be a smooth function and δ a gauge on $[\boldsymbol{a},\boldsymbol{b}]$. A triple $(\boldsymbol{\xi},\boldsymbol{I},\psi)$, with $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$ and $\boldsymbol{I} \subseteq [\boldsymbol{a},\boldsymbol{b}]$, is said to be δ*-fine* if

$$\text{supp } \psi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi};\delta(\boldsymbol{\xi})).$$

Note that $\boldsymbol{\xi}$ may not be in supp ψ. If $(\boldsymbol{\xi},\boldsymbol{I},\psi)$ is δ-fine and $\boldsymbol{x} \notin I$, then $\psi(\boldsymbol{x}) = 0$.

If δ_1 and δ_2 are gauges on $[\boldsymbol{a},\boldsymbol{b}]$ such that $\delta_1(\boldsymbol{\xi}) \geq \delta_2(\boldsymbol{\xi})$ and $(\boldsymbol{\xi},\boldsymbol{I},\psi)$ is δ_2-fine, then $(\boldsymbol{\xi},\boldsymbol{I},\psi)$ is also δ_1-fine.

Throughout this paper, a *division* of $[\boldsymbol{a},\boldsymbol{b}]$ is a finite collection $D=\{\boldsymbol{I}_k\}_{k=1}^m$ of subintervals $\boldsymbol{I}_k=\prod_{i=1}^n [a_i^{(k)}, b_i^{(k)}]$ of $[\boldsymbol{a},\boldsymbol{b}]$ such that $int(\boldsymbol{I}_k)\cap int(\boldsymbol{I}_j)=\varnothing$ for $k\neq j$ and $\bigcup_{k=1}^m \boldsymbol{I}_k=[\boldsymbol{a},\boldsymbol{b}]$. A division $D=\{\boldsymbol{I}_k\}_{k=1}^m$ of $[\boldsymbol{a},\boldsymbol{b}]=\prod_{k=1}^n [a_k,b_k]$ is a *net* if for each $k=1,2,\ldots,m$ there exists a division D_k of $[a_k,b_k]$ such that

$$D=\left\{\prod_{k=1}^m [s_k,t_k] : [s_k,t_k]\in D_k \ \text{ for } \ k=1,2,\ldots,m\right\}.$$

Definition 2.3 [2] A finite collection $D=\{(\boldsymbol{\xi}_k,\boldsymbol{I}_k,\psi_k)\}_{k=1}^m$ is said to be a *δ-fine partial division* of $[\boldsymbol{a},\boldsymbol{b}]$ if the collection $\{\psi_k\}_{k=1}^m$ is a partial partition of unity and every $(\boldsymbol{\xi}_k,\boldsymbol{I}_k,\psi_k)$ is δ-fine. If $\{\psi_k\}_{k=1}^m$ is a partition of unity, then D is said to be a *δ-fine division* of $[\boldsymbol{a},\boldsymbol{b}]$.

The existence of δ-fine divisions of $[\boldsymbol{a},\boldsymbol{b}]$ is guaranteed by the open covering theorem and the existence of a partition of unity [2].

Let $D=\{(\boldsymbol{\xi}_k,\boldsymbol{I}_k,\psi_k)\}_{k=1}^m$ be a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$, and $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ be a function. Suppose that for each $k\in\{1,2,\cdots,m\}$, the Riemann-Stieltjes integral $\int_{\boldsymbol{I}_k}\psi_k\, dg$ exists. Define the *PUL-Stieltjes sum of f with respect to g over D* by

$$S(f,g,D)=\sum_{k=1}^m f(\boldsymbol{\xi}_k)\int_{\boldsymbol{I}_k}\psi_k(\boldsymbol{x})\ dg(\boldsymbol{x})=\sum_{k=1}^m f(\boldsymbol{\xi}_k)\int_{\boldsymbol{I}_k}\psi_k\ dg.$$

For brevity, we write a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$ by $D=\{(\boldsymbol{\xi},\boldsymbol{I},\psi)\}$ and a PUL-Stieltjes sum of f with respect to g over D by

$$S(f,g,D)=(D)\sum f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\psi\ dg=\sum_{D} f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\psi\ dg.$$

Remark 2.4 *If $D_1=\{(\boldsymbol{\xi}_k,\boldsymbol{I}_k,\varphi_k)\}_{k=1}^m$ and $D_2=\{(\boldsymbol{\xi}_j,\boldsymbol{I}_j,\psi_j)\}_{j=1}^m$ are two δ-fine divisions of $[\boldsymbol{a},\boldsymbol{b}]$, then $S(f,g,D_1)=S(f,g,D_2)$.*

Proof: Let $D_1=\{(\boldsymbol{\xi}_k,\boldsymbol{I}_k,\varphi_k)\}_{k=1}^m$ and $D_2=\{(\boldsymbol{\xi}_j,\boldsymbol{I}_j,\psi_j)\}_{j=1}^m$ be δ-fine divisions of $[\boldsymbol{a},\boldsymbol{b}]$. Note that D_1 and D_2 differ only by the partition of unity. Then for each $k=1,2,\ldots,m$, $\psi_j(\boldsymbol{x})=0,\ \text{ for all } j\neq k \ \text{ and } \ \boldsymbol{x}\notin\boldsymbol{I}_k$. Thus, for each $k=1,2,\ldots,m$ and for all $\boldsymbol{x}\in\boldsymbol{I}_k$

$$\sum_{j=1}^m \psi_j(\boldsymbol{x})=\psi_k(\boldsymbol{x}).$$

Hence,

$$\begin{aligned}S(f,g,D_1)&=\sum_{k=1}^m f(\boldsymbol{\xi}_k)\int_{\boldsymbol{I}_k}\varphi_k(\boldsymbol{x})\ dg(\boldsymbol{x})=\sum_{k=1}^m f(\boldsymbol{\xi}_k)\int_{\boldsymbol{I}_k}1\cdot\varphi_k(\boldsymbol{x})\ dg(\boldsymbol{x})\\&=\sum_{k=1}^m f(\boldsymbol{\xi}_k)\int_{\boldsymbol{I}_k}\left[\sum_{j=1}^m\psi_j(\boldsymbol{x})\right]\cdot\varphi_k(\boldsymbol{x})\ dg(\boldsymbol{x})=\sum_{k=1}^m f(\boldsymbol{\xi}_k)\int_{\boldsymbol{I}_k}\psi_k(\boldsymbol{x})\cdot\varphi_k(\boldsymbol{x})\ dg(\boldsymbol{x}).\end{aligned}$$

Similarly,

$$S(f,g,D_2) = \sum_{j=1}^{m} f(\boldsymbol{\xi}_k) \int_{\boldsymbol{I}_j} \psi_j(\boldsymbol{x})\ dg(\boldsymbol{x}) = \sum_{j=1}^{m} f(\boldsymbol{\xi}_j) \int_{\boldsymbol{I}_j} 1 \cdot \psi_j(\boldsymbol{x})\ dg(\boldsymbol{x})$$
$$= \sum_{j=1}^{m} f(\boldsymbol{\xi}_j) \int_{\boldsymbol{I}_j} \left[\sum_{k=1}^{m} \varphi_k(\boldsymbol{x})\right] \cdot \psi_j(\boldsymbol{x})\ dg(\boldsymbol{x}) = \sum_{j=1}^{m} f(\boldsymbol{\xi}_j) \int_{\boldsymbol{I}_j} \varphi_j(\boldsymbol{x}) \cdot \psi_j(\boldsymbol{x})\ dg(\boldsymbol{x}).$$

Consequently, $S(f,g,D_1) = S(f,g,D_2)$. ■

Definition 2.5 Let $(X, \|\cdot\|)$ be a Banach space. A function $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ is said to be *PUL-Stieltjes integrable to* $A \in X$ with respect to $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ if for every $\epsilon > 0$, there exists a gauge $\delta(\boldsymbol{\xi})$ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for every δ-fine division $D = \{(\boldsymbol{\xi}_k, \boldsymbol{I}_k, \psi_k)\}_{k=1}^{m}$ of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\|S(f,g,D) - A\| < \epsilon.$$

If A is the PUL-Stieltjes integral of f with respect to g, then we write

$$A = \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg.$$

Note that Remark 2.4 means that a PUL-Stieltjes sum is independent of the choice of the partition of unity. Consequently, the value the PUL-Stieltjes integral is independent of the choice of the partition of unity.

Theorem 2.6 *The PUL-Stieltjes integral of f with respect to g is unique.*

Proof: Let A and B be PUL-Stieltjes integrals of f with respect to g. Then there is a gauge $\delta_1(\boldsymbol{\xi}) > 0$ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ_1-fine division D of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\|S(f,g,D) - A\| < \frac{\epsilon}{2}.$$

Similarly, there exists a gauge $\delta_2(\boldsymbol{\xi})$ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ_2-fine division D' of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\left\|S(f,g,D') - B\right\| < \frac{\epsilon}{2}.$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then δ is a gauge in $[\boldsymbol{a},\boldsymbol{b}]$. Now, fix a δ-fine division D of $[\boldsymbol{a},\boldsymbol{b}]$. Then D is both δ_1-fine and δ_2-fine. Thus,

$$\left\|A - B\right\| \leq \left\|A - S(f,g,D)\right\| + \left\|S(f,g,D) - B\right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\|A - B\| = 0$. Hence, $A = B$. ■

3 Simple Properties of PUL-Stieltjes Integral

Here, we show that the PUL-Stieltjes integral has the homogeneity and linearity properties over both the integrands and integrators.

Theorem 3.1 *If $f_1 : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $f_2 : [\boldsymbol{a}, \boldsymbol{b}] \to X$ are PUL-Stieltjes integrable with respect to $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $c \in \mathbb{R}$, then cf_1 and $f_1 + f_2$ is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} cf_1 \ dg = c\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg \quad \text{and} \quad \int_{[\boldsymbol{a},\boldsymbol{b}]} (f_1 + f_2) \ dg = \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg + \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2 \ dg.$$

Proof: Let $c \in \mathbb{R}$ and let $\epsilon > 0$. Then there exists a gauge $\delta(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| S(f, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg \right\| < \frac{\epsilon}{1 + |c|}.$$

Thus, for any δ-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$ we have

$$\left\| S(cf, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} cf_1 \ dg \right\| = |c| \cdot \left\| S(f, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg \right\| < |c| \cdot \frac{\epsilon}{1 + |c|} < \epsilon.$$

Hence, cf_1 is PUL-Stieltjes integrable with respect to g and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} cf_1 \ dg = c\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg.$$

For the remaining part, let $\epsilon > 0$. Then there exists a gauge $\delta_1(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ_1-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| S(f_1, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg \right\| < \frac{\epsilon}{2}.$$

Also, there exists a gauge $\delta_2(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ_2-fine division D' of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| S(f_2, g, D') - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2 \ dg \right\| < \frac{\epsilon}{2}.$$

Let $\delta(\boldsymbol{\xi}) = \min\{\delta_1(\boldsymbol{\xi}), \delta_2(\boldsymbol{\xi})\}$ for all $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$. Let D be a δ-fine of $[\boldsymbol{a}, \boldsymbol{b}]$. Then D is both δ_1-fine and δ_2-fine of $[\boldsymbol{a}, \boldsymbol{b}]$. Thus,

$$\begin{aligned} &\left\| S(f_1 + f_2, g, D) - \left[\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg + \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2 \ dg \right] \right\| \\ &\quad \leq \left\| S(f_1, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg \right\| + \left\| S(f_2, g, D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2 \ dg \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, $f_1 + f_2$ is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} (f_1 + f_2) \ dg = \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1 \ dg + \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2 \ dg. \qquad \blacksquare$$

Theorem 3.2 *If $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is PUL-Stieltjes integrable with respect to both $g_1 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ on $[\boldsymbol{a}, \boldsymbol{b}]$ and $c \in \mathbb{R}$, then f PUL-Stieltjes integrable with respect to both cg_1 and $g_1 + g_2$ on $[\boldsymbol{a}, \boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ d(cg_1) = c\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_1 \quad \textit{and} \quad \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ d(g_1+g_2) = \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_1 + \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_2.$$

The proof is similar to Theorem 3.1.

4 Cauchy Criterion

We the Cauchy criterion for the PUL-Stieltjes integral.

Theorem 4.1 (Cauchy Criterion) A function $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is PUL-Stieltjes integrable with respect to $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ on $[\boldsymbol{a}, \boldsymbol{b}]$ if and only if for any $\epsilon > 0$, there exists a gauge $\delta(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any pair of δ-fine divisions D_1 and D_2 of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\|S(f,g,D_1) - S(f,g,D_2)\| < \epsilon.$$

Proof: $(\Rightarrow)$ Let $\epsilon > 0$. Then there is a gauge $\delta(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| S(f,g,D) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg \right\| < \frac{\epsilon}{2}.$$

Let D_1 and D_2 be any two δ-fine divisions of $[\boldsymbol{a}, \boldsymbol{b}]$. Then

$$\begin{aligned} &\left\| S(f,g,D_1) - S(f,g,D_2) \right\| \\ &\qquad \le \left\| S(f,g,D_1) - \int_{[\boldsymbol{a},\boldsymbol{b}]} f dg \right\| + \left\| \int_{[\boldsymbol{a},\boldsymbol{b}]} f dg - S(f,g,D_2) \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$(\Leftarrow)$ By hypothesis, for each $n \in \mathbb{N}$, there exists a gauge $\delta_n(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ_n-fine divisions D_n and D_n' of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| S(f,g,D_n) - S(f,g,D_n') \right\| < \frac{1}{n}. \tag{1}$$

We may assume that $\delta_n(\boldsymbol{\xi}) \ge \delta_{n+1}(\boldsymbol{\xi})$ for each $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$ and for all $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, let D_n be a fix δ_n-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ and consider its corresponding PUL-Stieltjes sum $s_n = S(f,g,D_n)$. We show that the sequence $\langle s_n \rangle_{n=1}^{+\infty}$ is Cauchy in X. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ with $\frac{1}{N} < \epsilon$. Suppose that $n, m \ge N$. Then $\delta_n(\boldsymbol{\xi}) \le \delta_N(\boldsymbol{\xi})$ and $\delta_m(\boldsymbol{\xi}) \le \delta_N(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$. Thus, D_n and D_m are both δ_N-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Hence,

$$\|s_n - s_m\| = \left\| S(f,g,D_n) - S(f,g,D_m) \right\| < \frac{1}{N} < \epsilon.$$

This shows that $\langle s_n \rangle_{n=1}^{+\infty}$ is Cauchy in X. Since X is complete, $\langle s_n \rangle_{n=1}^{+\infty}$ converges in X, say, $\lim_{n\to\infty} s_n = A$.

We now show that f is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \; dg = A.$$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} s_n = A$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|S(f, g, D_n) - A\| = \|s_n - A\| < \frac{\epsilon}{2}. \tag{2}$$

We chose $N \in \mathbb{N}$ in which $\frac{1}{N} < \epsilon$. Put $\delta(\boldsymbol{\xi}) = \delta_N(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$. Let D be any δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then D is δ_N-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Since $N \geq N_2$, by (1) we have

$$\|S(f, g, D) - S(f, g, D_N)\| < \frac{1}{N} < \frac{\epsilon}{2}. \tag{3}$$

Also, since $N \geq N_1$, inequality (2) for $n = N$; i.e.

$$\|S(f, g, D_N) - A\| < \frac{\epsilon}{2}. \tag{4}$$

Hence, by (3) and (4) we have

$$\|S(f, g, D) - A\| \leq \|S(f, g, D) - S(f, g, D_N)\| + \|S(f, g, D_N) - A\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that f is a PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \; dg = A. \qquad \blacksquare$$

In what follows, we denote the set of all compact subintervals of $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{R}^n$ by $\mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$.

Corollary 4.2 If $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is PUL-Stieltjes integrable with respect to $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ on $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{I} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$, then f is PUL-Stieltjes integrable with respect to g on $\boldsymbol{I}$.

Proof: Let $\epsilon > 0$. By Theorem 4.1, there exists a gauge $\delta(\boldsymbol{\xi})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ-fine divisions D_1 and D_2 of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\|S\,(f, g, D_1) - S\,(f, g, D_2)\| < \epsilon. \tag{5}$$

If $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$, then we are done. Suppose that $\boldsymbol{I} \subset [\boldsymbol{a}, \boldsymbol{b}]$. Then there is a finite collection $\mathcal{J} \subseteq \mathcal{I}([\boldsymbol{a}, \boldsymbol{b}])$ such that $\boldsymbol{I} \notin \mathcal{J}$ and $\mathcal{J} \cup \{\boldsymbol{I}\}$ is a net of $[\boldsymbol{a}, \boldsymbol{b}]$. For each $\boldsymbol{J} \in \mathcal{J} \cup \{\boldsymbol{I}\}$, fix a δ-fine division $D_{\boldsymbol{J}}$ of $\boldsymbol{J}$. Let $D_{\boldsymbol{I}}^{(1)}$ and $D_{\boldsymbol{I}}^{(2)}$ be two δ-fine divisions of $\boldsymbol{I}$. Let

$$D_1 = D_{\boldsymbol{I}}^{(1)} \cup \bigcup_{\boldsymbol{J} \in \mathcal{J}} D_{\boldsymbol{J}} \quad \text{and} \quad D_2 = D_{\boldsymbol{I}}^{(2)} \cup \bigcup_{\boldsymbol{J} \in \mathcal{J}} D_{\boldsymbol{J}}.$$

Then D_1 and D_2 are δ-fine divisions of $[\boldsymbol{a},\boldsymbol{b}]$ and

$$S(f,g,D_{\boldsymbol{I}}^{(1)}) = S(f,g,D_1) - \sum_{\boldsymbol{J}\in\mathcal{J}} S(f,g,D_{\boldsymbol{J}}) \quad \text{and} S(f,g,D_{\boldsymbol{I}}^{(2)}) = S(f,g,D_2) - \sum_{\boldsymbol{J}\in\mathcal{J}} S(f,g,D_{\boldsymbol{J}}).$$

Thus, by (5)

$$\left\|S(f,g,D_{\boldsymbol{I}}^{(1)}) - S(f,g,D_{\boldsymbol{I}}^{(2)})\right\| = \|S(f,g,D_1) - S(f,g,D_2)\| < \epsilon.$$

The desired result now follows from Theorem 4.1. ■

In what follows, let $\mathcal{V}[\boldsymbol{u},\boldsymbol{v}]$ be the collection of all the vertices of an interval $[\boldsymbol{u},\boldsymbol{v}]$.

Definition 4.3 [2] Let $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$. The *total variation of g over* $[\boldsymbol{a},\boldsymbol{b}]$ is given by

$$Var(g,[\boldsymbol{a},\boldsymbol{b}]) = \sup\left\{\sum_{[\boldsymbol{u},\boldsymbol{v}]\in D} |\Delta_g([\boldsymbol{u},\boldsymbol{v}])| : \ D \text{ is a division of} [\boldsymbol{a},\boldsymbol{b}]\right\}$$

where $\Delta_g([\boldsymbol{u},\boldsymbol{v}]) = \displaystyle\sum_{\boldsymbol{t}\in\mathcal{V}[\boldsymbol{u},\boldsymbol{v}]} g(\boldsymbol{t}) \prod_{k=1}^{m} (-1)^{\chi_{\{u_k\}}(t_k)}$.

Definition 4.4 [8] A function $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ is said to be of *bounded variation* on $[\boldsymbol{a},\boldsymbol{b}]$ if $Var(g,[\boldsymbol{a},\boldsymbol{b}])$ is finite. We denote $BV([\boldsymbol{a},\boldsymbol{b}])$ to be the collection of functions of bounded variation on the interval $[\boldsymbol{a},\boldsymbol{b}]$.

Theorem 4.5 *If $f:[\boldsymbol{a},\boldsymbol{b}]\to X$ is continuous on $[\boldsymbol{a},\boldsymbol{b}]$ and $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ is of bounded variation on $[\boldsymbol{a},\boldsymbol{b}]$, then f is PUL-stieltjes integrable on $[\boldsymbol{a},\boldsymbol{b}]$ with respect to g.*

Proof: Let $\epsilon > 0$. Note that the Riemann-stieltjes integral $\displaystyle\int_{[\boldsymbol{a},\boldsymbol{b}]} \varphi dg$ exists, whenever φ is a partition of unity and g is of bounded variation on $[\boldsymbol{a},\boldsymbol{b}]$. Since $g\in BV([\boldsymbol{a},\boldsymbol{b}])$, $M = V(g,[\boldsymbol{a},\boldsymbol{b}])\in\mathbb{R}$. Since f is continuous on $[\boldsymbol{a},\boldsymbol{b}]$, then f is uniformly continuous on $[\boldsymbol{a},\boldsymbol{b}]$. Thus, there exists a $\delta>0$ such that for any $\boldsymbol{x},\boldsymbol{y}\in[\boldsymbol{a},\boldsymbol{b}]$ with $\|\boldsymbol{x}-\boldsymbol{y}\|_{\mathbb{R}^n} < \delta(\boldsymbol{x})$, we have

$$\|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_X < \frac{\epsilon}{2[M+1]}.$$

Let $D_1 = \{(\boldsymbol{\xi},\boldsymbol{I},\varphi)\}$ and $D_2 = \{(\boldsymbol{\zeta},\boldsymbol{J},\psi)\}$ be any two δ-fine divisions of $[\boldsymbol{a},\boldsymbol{b}]$. Let $D_3 = \{\boldsymbol{\gamma},\boldsymbol{K},\sigma\}$ be a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$, where $\boldsymbol{K} = \boldsymbol{I}\cap\boldsymbol{J}$ with $\boldsymbol{I}\in D_1$ and $\boldsymbol{J}\in D_2$. . Observe that

$$S(f,g,D_1) = \sum_{\boldsymbol{I}\in D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi dg = \sum_{\boldsymbol{I}\in D_1} f(\boldsymbol{\xi}) \left[\sum_{\boldsymbol{J}\in D_2} \int_{\boldsymbol{I}\cap\boldsymbol{J}} \varphi dg\right] = \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\xi}) \int_{\boldsymbol{K}} \sigma dg \quad \text{and}$$

$$S(f,g,D_2) = \sum_{\boldsymbol{J}\in D_2} f(\boldsymbol{\zeta}) \int_{\boldsymbol{J}} \psi dg = \sum_{\boldsymbol{J}\in D_2} f(\boldsymbol{\zeta}) \left[\sum_{\boldsymbol{I}\in D_1} \int_{\boldsymbol{J}\cap\boldsymbol{I}} \psi dg\right] = \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\zeta}) \int_{\boldsymbol{K}} \sigma dg.$$

Then

$$\begin{aligned}
&\left\|S(f,g,D_1)-S(f,g,D_2)\right\| \le \left\|S(f,g,D_1)-S(f,g,D_3)\right\| + \left\|S(f,g,D_3)-S(f,g,D_2)\right\| \\
&= \left\| \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\xi}) \int_{\boldsymbol{K}} \sigma dg - \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\gamma}) \int_{\boldsymbol{K}} \sigma dg \right\| + \left\| \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\gamma}) \int_{\boldsymbol{K}} \sigma dg - \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\zeta}) \int_{\boldsymbol{K}} \sigma dg \right\| \\
&\le \sum_{\boldsymbol{K}\in D_3} \left[\left\| f(\boldsymbol{\xi}) - f(\boldsymbol{\gamma}) \right\| \cdot \left| \int_{\boldsymbol{K}} \sigma dg \right| \right] + \sum_{\boldsymbol{K}\in D_3} \left[\left\| f(\boldsymbol{\gamma}) - f(\boldsymbol{\zeta}) \right\| \cdot \left| \int_{\boldsymbol{K}} \sigma dg \right| \right] \\
&< \sum_{\boldsymbol{K}\in D_3} \left[\frac{\epsilon}{2[M+1]} \cdot \left| \int_{\boldsymbol{K}} \sigma dg \right| \right] + \sum_{\boldsymbol{K}\in D_3} \left[\frac{\epsilon}{2[M+1]} \cdot \left| \int_{\boldsymbol{K}} \sigma dg \right| \right] \\
&= \frac{\epsilon}{M+1} \cdot \sum_{\boldsymbol{K}\in D_3} \left| \int_{\boldsymbol{K}} \sigma dg \right| = \frac{\epsilon}{M+1} \cdot \sum_{\boldsymbol{K}\in D_3} \Delta_g(\boldsymbol{K}) \le \frac{\epsilon}{M+1} \cdot M < \epsilon.
\end{aligned}$$

Therefore, f is PUL-Stieltjes integrable on $[\boldsymbol{a},\boldsymbol{b}]$ with respect to g. ■

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