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On the Absolute Euler Summability of a Factored Fourier Series

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Abstract

In this paper I have proved a theorem on the Absolute Euler Summability of a Factored Fourier series, Which generalizes various known results. However the theorem is as follows

Theorem :- If

$$\phi(t) = \int_0^t |\phi(t)|\, dt = O\left(\frac{t}{\left(\log\left(\frac{1}{t}\right)\right)^{\alpha}}\right) \quad \alpha > \frac{1}{2}$$

then $\sum_{n=1}^{\infty} A_n(x)\,\lambda_n \in |E,q|, \quad q > 0$

where λ_n is convex sequence such that $\sum \frac{\lambda_n}{n}$ is convergent.

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Introduction

1. Definitions and Notations :

Definition 1 – Let $\sum a_n$ be an infinite series with the sequence of partial sums $\{s_n\}$. The Euler's series-to-series transform

$$b_n = (q+1)^{-n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} a_\upsilon \quad q>0,$$

If $\sum b_n$ is convergent, we say that $\sum a_n$ (or the sequence $\{s_n\}$) is summable (E,q), q > 0, in short we write

$$\sum a_n \in (E,\ q) \left[\text{or } \{s_n\} \in |E, q|\right],\ q>0$$

If $\sum b_n$ is absolutely convergent, we say that $\sum a_n$ (or the sequence $\{s_n\}$) is summable (E, q), q > 0, in short we write[1-2]

$$\sum a_n \in |E,\ q| \quad \left[\text{or } \{s_n\} \in |E, q|\right],\ q>0$$

Definition 2 : The series $\sum a_n$ is said to be summable $|E_\alpha|$ $(0<\alpha<1)$, if

$$t_n = \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} \alpha^\upsilon (1-\alpha)^{n-\upsilon} s_\upsilon$$

(1.1) and $\sum |t_n - t_{n-1}| < \infty$

Since

$$\tau_n = \sum_{\upsilon=1}^{n} \binom{n}{\upsilon} \alpha^\upsilon (1-\alpha)^{n-\upsilon} \upsilon a_\upsilon$$

$$= n\,(t_n - t_{n-1})$$

(1.1) is equivalent to

$$(1.2) \qquad \sum \frac{|\tau_n|}{n} < \infty$$

It is easy to notice Kwee[3] that definition 1 reduces to definition 2, with the substitution $q = \dfrac{(1-\alpha)}{\alpha}$, $0<\alpha<1$,

i.e.

$$\left| E, \frac{(1-\alpha)}{\alpha} \right| \sim |E_\alpha|, \qquad 0<\alpha<1.$$

It is also known from Kwee[3] and Tripathy[6] that the E_α means given above can be derived from Hausdorff means.

Let f (t) be Lebesgue integrable over (-π,π) and periodic with period 2π and let

$$(1.3)\ f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

We write throughout

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \quad \alpha>0,$$

$$\phi_0(t) = \phi(t)$$

$$\Phi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \phi_\alpha(t), \qquad (\alpha \ge 0)$$

$$\Delta \mu_n = \mu_n - \mu_{n+1}$$

We need the following estimates for the proof of the theorem

$$(1.4)\ g_n(t) = \begin{cases} O(n^{\beta}) \\ O(t^{1} n^{\beta-1}) + O(n^{\beta} \rho^{n}(t)), & \beta \neq 0 \end{cases}$$

and

$$J_n(t) = \begin{cases} O(n^{(\alpha-2)/2}) \\ O(n^{(\alpha-2)/2}\ \rho^{n}(\delta)) + O(n^{(\alpha-2)/2}) \end{cases}$$

for $0 < \alpha < 2$ and $0 < \delta < \pi$

2. The absolute Euler summability of a Fourier series have been studied by Mohanty and Mohapatra[4] and Kwee[3] independently and their results read as follows.

Theorem A : If $\phi(t) \log\left(\frac{1}{t}\right) \in BV(0, \delta)$ $0 < \delta < 1$, then

$$\sum_{n=0}^{\infty} A_n(x) \in |E, q|, \ q > 0$$

We state below a result on absolute Euler summability of a factored Fourier series due to Tripathy[6].

Theorem B: If $\phi_{\alpha}(t) \in BV(0, \pi)$, $0 < \alpha < \frac{3}{2}$

then $\sum_{n=1}^{\infty} n^{-\lambda} A_n(x) \in |E, q|, \quad q > 0,$

where $\lambda > \max\left(\alpha - \frac{1}{2}, \frac{1}{2}\right)$.

Ray and Patra[5] extended the above theorem by proving the following theorem.

Theorem C : If $\phi^{\alpha}(t) \in BV\ (0, \delta), \delta > 0,$

then $\sum A_n(x)/n^{\alpha/2} \in |E, q|, \quad q > 0$

for $0 < \alpha \leq 2.$

The object of this paper is to generalize above theorems by establishing the following theorem.

3. We assert the following main theorem.

Theorem : If

$$\phi(t) = \int_0^t |\phi(t)|\, dt = O\left(\frac{t}{\left(\log\left(\frac{1}{t}\right)\right)^{\alpha}}\right) \quad \alpha > \frac{1}{2}$$

then $\sum_{n=1}^{\infty} A_n(x)\, \lambda_n \in |E, q|, \quad q > 0$

where λ_n is convex sequence such that $\sum \frac{\lambda_n}{n}$ is convergent[4].

4. Proof of the theorem : Let $\tau_n(x)$ represents the $|E, q|$ mean of the sequence $\{\lambda_\upsilon A_\upsilon(x)\}$. Then we haves

$$\tau_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t)\, g_n(t)\, dt$$

$$= \frac{2}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\pi}\right) \phi(t)\, g_n(t)\, dt$$

$$= I_1 + I_2, \quad \text{say}$$

To prove the theorem, it is sufficient to prove that

$$\sum_{n=2}^{m}\frac{|\tau_n|}{n}=\sum_{n=2}^{m}\frac{|I_1|}{n}+\sum_{n=2}^{m}\frac{|I_2|}{n}<\infty\quad(m\to\infty).$$

$$=\sum{}_1+\sum{}_2,\ \text{say}$$

Now

$$\sum{}_1=K\sum_{n=2}^{m}\left(\frac{1}{n^2}\right)\left(\int_0^{1/n}\phi|\ (t)\,|\,n^{\beta}\,dt\sum_{\upsilon=1}^{n-1}\Delta\lambda_\upsilon\,\upsilon^2\right)$$

$$+K\sum_{n=2}^{m}\lambda_n\int_0^{1/n}|\,\phi(t)\,|\,t^{-1}\,n^{\beta-1}\,dt$$

$$=K\sum_{n=2}^{m}\left(\frac{n^{\beta}}{n^3}\right)\sum_{\upsilon=1}^{n-1}\Delta\lambda_\upsilon\,\upsilon^2+K\sum_{n=2}^{m}\frac{n^{\beta-1}\lambda_n}{n}$$

$$=K\sum_{\upsilon=2}^{\infty}\Delta\lambda_\upsilon\,\upsilon^2\sum_{n=\upsilon+1}^{\infty}n^{\beta-3}+K$$

$$=K$$

where K is an arbitrary constant.

Again

$$\sum{}_2\leq K\sum_{n=2}^{m}\frac{\lambda_n}{n}\left|\int_{1/n}^{\pi}\phi(t)\,J_n(t)\,dt\right|$$

$$+K\sum_{n=2}^{m}\frac{1}{n}\int_{1/n}^{\pi}|\,\phi(t)|\,dt\ \left\{O\left(n^{(\alpha-2)/2}\right)\right\}$$

$$\left\{+O\left(n^{(\alpha-2)/2}\,\rho^n(\delta)\right)+O\left(n^{(\alpha-2)/2}\,\lambda_n\right)+O\left(\frac{1}{n}\sum_{\upsilon=1}^{n-1}\Delta\lambda_\upsilon\,\upsilon\right)\right\}$$

$$\leq K\sum_{n=2}^{m}\frac{\lambda_n}{n}\left|\int_0^{\pi}\phi(t)\,J_n(t)\,dt\right|+K\sum_{n=2}^{m}\frac{\lambda_n}{n}\left|\int_0^{1/n}\phi(t)\,J_n(t)\,dt\right|$$

$$+K\sum_{n=2}^{m}\frac{n^{\alpha}}{n^2}\int_{1/n}^{\pi}|\,\phi(t)|\,dt+K\sum_{n=2}^{m}\frac{\lambda_n\,n^{(\alpha-2)/2}}{n^2}\int_{1/n}^{\pi}|\,\phi(t)|\,dt$$

$$+K\sum_{n=2}^{m}\frac{\lambda_n}{n^2}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^2}dt \quad +K\sum_{n=2}^{m}\left(\frac{1}{n}\right)\sum_{\upsilon=1}^{n-1}\Delta\lambda_\upsilon\,\upsilon\int_{1/n}^{\pi}\frac{|\phi(t)|}{t}dt$$

$$=K\sum_{n=2}^{m}(1-\alpha)^n\frac{\lambda_n}{n}\left|s_n(x)-f(x)\right|+K\sum_{n=2}^{m}\frac{\lambda_n}{n}$$

$$+K\sum_{n=2}^{m}\frac{n^\alpha}{n^2}+K\sum_{n=2}^{m}\frac{\lambda_n n^{(\alpha-2)/2}}{n^2}$$

$$+K\sum_{n=2}^{m}\frac{\lambda_n}{n}+K\sum_{n=2}^{m}\left(\frac{1}{n}\right)(1-\alpha)^n\sum_{\upsilon=1}^{n-1}\Delta\lambda_\upsilon\,\upsilon$$

$$=K\sum_{n=2}^{m}\frac{(1-\alpha)^n\lambda_n}{n}\left|s_n(x)-f(x)\right|+K\sum_{\upsilon=2}^{\infty}\frac{\Delta\lambda_\upsilon\,\upsilon(1-\alpha)^\upsilon}{\upsilon}+K$$

$$=K\sum_{n=2}^{m}\frac{(1-\alpha)^n\lambda_n}{n}\left|s_n(x)-f(x)\right|+K$$

using the Abel's transformation

$$=K\sum_{n=2}^{m-1}\left(\sum_{\upsilon=2}^{n}\left|s_\upsilon(x)-f(x)\right|\right)(1-\alpha)^n\,\Delta\left(\frac{\lambda_n}{n}\right)$$

$$+\frac{\lambda_m}{m}\sum_{n=2}^{m}\left|s_n(x)-f(x)\right|(1-\alpha)^n+K$$

$$=K\sum_{n=2}^{m-1}(1-\alpha)^n\,\Delta\left(\frac{\lambda_n}{n}\right)O(n)\;+(1-\alpha)^m\frac{\lambda_m}{m}O(m)+K$$

$$=O(1)$$

This completes the proof of the theorem.

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