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Some Common Fixed Point Theorem in Cone Metric Space using Rational Inequality

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Abstract

In this paper we prove the common fixed point theorem in cone metric space for rational inequality in normal cone setting. Our results generalize the main result of Jaggi[5] and Dass, Gupta[1].

Key words: Common fixed point theorem, Cone Metric Space, Rational Inequality.

Introduction

The Banach contraction principle with rational expressions have been expanded and some common fixed point theorems have been obtained[1,2]. Huang and Zhang[4] initiated cone metric spaces, which is a generalization of metric spaces, by substituting the real numbers with ordered Banach spaces. They have considered convergence in cone metric spaces, introduced completeness of cone metric spaces, and proved a Banach contraction mapping theorem, and some other fixed point theorems involving contractive type mappings in cone metric spaces using the normality condition. Later, various authors have proved some common fixed point theorems with normal and non – normal cones in these spaces[7,8]. Quite recently Mohammad Arshad *et al.*[6] have introduced almost Jaggi and Gupta contraction in partially ordered metric space to prove the fixed point theorem. In this paper we prove the common fixed point theorem in cone metric space for rational inequality in normal cone setting. Our results generalize the main result of jaggi[5] and Dass, Gupta[1].

Preliminaries:

Definition: Let E be a real Banach space and P a subset Of E. P is called a cone if and only if

(i) P is a closed, nonempty, and $P \neq \{0\}$;

(ii) $a, b \in R$, $a, b \geq o$, $x, y \in P \Rightarrow ax + by \in P$

(iii) $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{int } P$, denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq M \|y\|$.

The least positive number satisfying above is called the normal constant of P. In the following we always suppose E is a Banach space, P is a cone in E with int $P \neq \phi$ and $\leq$ is partial ordering with respect to P.

Definition: Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$:

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (x, d) is called a cone metric space.

Example: Let $E = R^2$, $P = \{ (x, y) \in E | x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ convergences to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \in N$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $o << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

Definition : Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.

Main Result:

Theorem: Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let $S, T : X \to X$ be an almost Jaggi contraction, for all $x, y \in X$ and

$$d(Sx, Ty) \leq \frac{\alpha\, d(Sx,\ Ty),\ d(x, Sx)}{d(x, y)} + \beta\, d(x, Sx) + L \min \{ d(x, Ty), d(y, Sx)\}$$

Where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.

Proof: Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_2 = Sx_1$

$$x_n = Tx_{n-1}, \ x_{n+1} = Sx_n$$

$$d(x_n, x_{n+1}) = d(Sx_{n-1}, Tx_n)$$

$$\leq \frac{\alpha\, d(Sx_{n-1},\ Tx_n),\ d(x_{n-1}, Sx_{n-1})}{d(x_{n-1}, x_n)} + \beta\, d(x_{n-1}, Sx_{n-1}) + L \min \{d(x_{n-1}, Tx_n), d(x_n, Sx_{n-1})\}$$

$$\leq \frac{\alpha\, d(x_n,\ x_{n+1}),\ d(x_{n-1}, x_n)}{d(x_{n-1}, x_n)} + \beta\, d(x_{n-1}, x_n) + L \min \{ d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}$$

$$(1 - \alpha)\, d(x_n, x_{n-1}) \leq \beta\, d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{\beta}{(1 - \alpha)}\, d(x_{n-1}, x_n)$$

$$k = \frac{\beta}{(1 - \alpha)}, \ \alpha + \beta < 1, \ 0 < k < 1$$

and by induction,

$$d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$$

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$$d(x_n, x_{n-1},) \leq d(x_{n-1},, x_{n-1},) + d(x_{n-1}, x_{n+2}) + \dots$$

$$+ d(x_{n+m-1}, x_m)$$

$$\leq (k^n + k^{n+1} + \dots + k^{n+m-1})\, d(x_0, x_1)$$

We get $\|d(x_n, x_m)\| \leq M \frac{k^n}{1-k} \|d(x_0, x_1)\|$ which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence x_n is a Cauchy sequence, so by completeness of X this sequence must be convergent in X.

$$d(u, Tu) \leq d(u, x_{n-1}) + d(x_{n-1}, Tu)$$

$$\leq d(u, x_{n+1}) + d(Sx_n, Tu)$$

$$\leq d(u, x_{n+1}) + \frac{\alpha\, d(Sx_n,\ Tu),\ d(x_n, Sx_n)}{d(x_n, u)} + \beta\, d(x_n, Sx_n)$$

$$+ L \min\{ d(x_n, Tu), d(u, Sx_n)\}$$

$$\leq d(u, x_{n+1}) + \frac{\alpha\, d(x_{n+1},\ u),\ d(x_n, x_{n+1})}{d(x_n, u)} + \beta\, d(x_n, x_{n+1})$$

$$+ L \min\{ d(x_n, u), d(u, x_{n+1})\}$$

$$\leq d(u, x_{n+1}) + \beta\, d(x_n, x_{n+1})$$

$$\|d(u, Tu)\| \leq \|d(u, x_{n+1})\| + \beta \|d(x_n, x_{n+1})\|$$

As $n \to 0$ we have $\|d(u, Tu)\| \leq 0$. Hence we get $u = Tu$. U is a fixed point of T.

Corollary 1: Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let *S*, T: X→ X be a Jaggi constraction

$$d(Sx, Ty) \leq \frac{\alpha\, d(Sx,\ Ty),\ d(x, Sx)}{d(x, y)} + \beta\, d(x, Sx)$$

for all x, y ∈ X and α, β ∈ [0, 1) with α + β < 1. Then S, T has a unique fixed point in X.

Proof : Set L = 0 in above theorem we get the required condition.

Corollary 2: Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let T: X → X be a Jaggi constraction

$$d(Tx, Ty) \leq \frac{\alpha\, d(Tx,\ Ty),\ d(x, Tx)}{d(x, y)} + \beta\, d(x, Tx)$$

for all x, y ∈ X and α, β ∈ [0, 1) with α + β < 1. Then T has a unique fixed point in X.

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