



ISSN 2231-346X

(Print)

JUSPS-A Vol. 29(5), 192-195 (2017). Periodicity-Monthly

Section A

(Online)



ISSN 2319-8044



Estd. 1989

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
 An International Open Free Access Peer Reviewed Research Journal of Mathematics
 website:- www.ultrascientist.org

Combinatorial Approach in Counting the Balanced Bicliques in the Join and Corona of Graphs

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<http://dx.doi.org/10.22147/jusps-A/290501>

Acceptance Date 27th March, 2017, Online Publication Date 2nd May, 2017

Abstract

In this paper, we characterized the balanced bicliques of graphs resulting from the join and the corona of graphs. Furthermore, we established the balanced biclique polynomials of the resulting graphs using combinatorial approach.
Key words: Biclique, balanced biclique polynomials, join and corona of graphs.

Mathematics Subject Classification (2000) MSC 05C15, 05C90

Introduction

An induced subgraph H of a graph G is a balanced biclique of G if $H \equiv K_{i,i}$ for some $i \in \{1, 2, \dots, \frac{V(G)}{2}\}$. In this case, the order of H is exactly $2i$. The balanced biclique polynomial of G is given by

$$b(G, x) = \sum_{i=1}^{\beta(G)} b_i(G) x^{2i}, \tag{1}$$

where $b_i(G)$ is the number of balanced bicliques of G of order $2i$ and $\beta(G)$ is the order of a maximum balanced biclique of G . Note that $b_1(G) = |E(G)|$ and $b_2(G)$ is just the number of induced C_4 of G . If G is a tree, then $\beta(G) = 2$. In this case, $b(G, x) = (n - 1)x^2$ since a tree has $n - 1$ edges. Hence, it would be interesting to consider cyclic graphs.

The following remark relates the maximum vertex degree of a graph $\Delta(G)$ and the degree of the polynomial.

Remark 1. The degree of the balanced biclique polynomial is at most $2\Delta(G)$. Moreover, if G has no induced C_4 and $\Delta(G) = 2$, then $\deg(G, x) = 2$.

We established results on the join and the corona of graphs.

Results

Join of Graphs :

In this section, we characterized the bicliques of the graphs resulting from the join of two connected graphs. Furthermore, we established the explicit form of the balanced biclique polynomial of the join of graphs.

The following definition describes how to construct the join $G \oplus H$ from the graphs G and H .

Definition 1. The join $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G \oplus H) = V(G) \cup V(H)$ and edge set $E(G \oplus H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

The fan F_9 is the join of the trivial graph K_1 and the path P_9 .

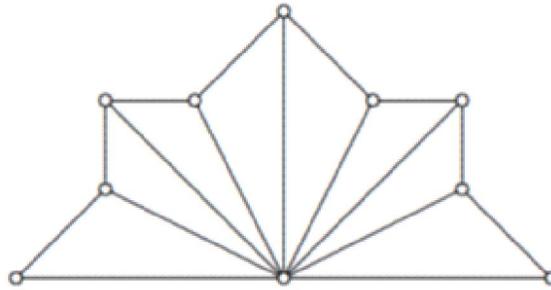


Figure 1. The Fan F_9

Lemma 1. A subset S of $V(G \oplus H)$ induces a balanced biclique in $G \oplus H$ if and only if it satisfies one of the following conditions:

1. S induces a balanced biclique in G .
2. S induces a balanced biclique in H .
3. $S = S_G \cup S_H$, where S_G is an independent set in G and S_H is an independent set in H with $|S_G| = |S_H|$.

Proof. Let S be a subset of $V(G \oplus H)$ such that S induces a balanced biclique in $G \oplus H$. Then $S = S_1 \cup S_2$ where $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$. Consider the following cases:

Case 1. $S_2 = \emptyset$. In this case, $S = S_1 \subseteq V(G)$. Hence, S induces a balanced biclique in G . Thus, condition (1) satisfied.

Case 2. $S_1 = \emptyset$. Here, $S = S_2$ and hence, S induces a balanced biclique in H . Consequently, S satisfies condition (2).

Case 3. $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. If S_1 is not independent, then there exist $u, v \in S_1$ such that $uv \in E(G)$. In this case, $[u, v, w, u]$ is a triangle in $\langle S_1 \cup S_2 \rangle$. Thus, S_1 must be independent. Similarly, S_2 must be independent.

Consequently, S satisfies condition (3) of Lemma 1.

The converse follows immediately from the definition of balanced bicliques of a graph. □
 The next result establishes the balanced biclique polynomial of the join of graphs.

Theorem 1. Let G and H be nontrivial connected graphs. Then,

$$b(G \oplus H, x) = b(G, x) + b(H, x) + \sum_{r=1}^{i(G)} \sum_{s=1}^{i(H)} \delta_{rs} i_r(G) j_s(H) x^{r+s} \tag{2}$$

Proof: The first term follows from the condition (1) of Lemma (1). The second term follows from the condition (2) of the above lemma. Now for each independent set S_G of G of cardinality k , a biclique is formed if and only if H has an independent set S_H of same cardinality. It is the pointwise product of the polynomials of the form $\langle i_1(G), i_2(G), i_3(G), \dots \rangle$ and $\langle i_1(H), i_2(H), i_3(H), \dots \rangle$, where $i_s(G)$ is the number of independent subsets of $V(G)$ of cardinality s for every s and $i_t(H)$ is the number of independent subsets of $V(H)$ of cardinality t for every t . Consequently, we have the last term of the polynomial. □

The next result gives the characterization of the balanced bicliques in the corona of graphs.

Corona of Graphs :

In this section, we characterize the balanced bicliques in the corona of the two nontrivial connected graphs. Moreover, we establish the balanced biclique polynomial of the graph resulting from the corona $G \circ H$.

Definition 2. The corona $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H , and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H .

Let us first illustrate the definition of the corona of two graphs. For this, consider the path P_3 and the complete graph K_3 .

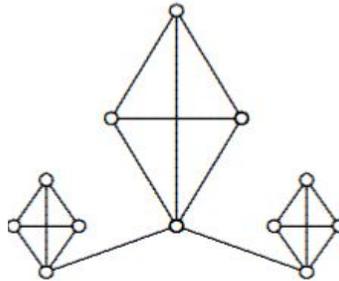


Figure 2. The corona $P_3 \circ K_3$.

Lemma 2. A subset S of $V(G \circ H)$ induces a balanced biclique in $G \circ H$ if and only if it satisfies on the following conditions:

1. S induces a balanced biclique of G .
2. S induces a balanced biclique in a copy of H .
3. $S = \{u, v_u\}$, where $u \in V(G)$ and v_u is a vertex of H_u , a copy of H attached to u .

Proof: Let S be a subset of $V(G \circ H)$ such that S induces a balanced biclique of $G \circ H$. Then $S = S_1 \cup S_2$ where $S_1 \subseteq V(G)$ and $S_2 \subseteq \cup_{u \in V(G)} H_u$. If $S_2 = \emptyset$, then S induces a balanced biclique in G . Hence, condition (1) is satisfied. If $S_1 = \emptyset$, then $S \subseteq \cup_{u \in V(G)} H_u$. Note that for every $(u, v) \in V(G) \times V(G)$, H_u

and H_v are independent. That is, H_u and H_v are not connected by an edge. Consequently, $S = H_w$ for some $w \in V(G)$. Accordingly, S induces a balanced biclique in a copy H_w of H attached to w . Thus, S induces a balanced biclique in a copy of H . Hence, condition (2) is satisfied. Lastly, suppose $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Then, there exist $u \in V(G)$ such that $u \in S_1$ and there exist $y \in S_2$ for some $y \in V(H_w)$. $w \in V(G)$ and H_w is a copy of H attached to w . Consequently, $S_2 \subseteq V(H_w)$ by the independence of each copy of H . This forces $u = w$. Thus, $ux \in E(G \circ H)$ for every $x \in H_u$. This implies that $|S_2| = 1$. Hence, $S = \{u, v_u\}$, where $v_u \in H_u$. Thus condition (3) is satisfied. The converse is immediate from the definition of bicliques. \square

Theorem 2. Let G and H be connected nontrivial graphs. Then

$$b(G \circ H, x) = b(G, x) + |V(G)|b(H, x) + |V(G)||V(H)|x^2 \quad (3)$$

Proof: The first term follows from the condition (1) of Lemma 2. Since a biclique in a copy of H is also a biclique in $G \circ H$, by condition (2) of the above Lemma, the second term follows from the construction of the corona of graphs. The last term of the polynomial follows from condition (3) of the above result. \square

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