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Path-Induced Geodetic Numbers of Some Graphs

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Abstract

This study introduces a new geodetic invariant, the path-induced geodetic number of a connected simple graph G. We investigate its properties and characterize the path-induced geodetic sets of some common graphs. Also, the path-induced geodetic numbers of these graphs are determined.

Key words: geodesic; geodetic set; geodetic number; path-induced geodetic set; path-induced geodetic number

2010 Mathematics Subject Classification: 05C12

1 Introduction

The concept on path-induced geodetic numbers of graphs follows from the definition of geodetic numbers of graphs introduced by Buckley and Harary in[2].

Let G be a connected simple nontrivial graph. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path joining u and v. A shortest u-v path is called a u-v geodesic. For every two vertices u and v of G, the interval $I_G[u, v]$ denotes all vertices lying in some u-v geodesic. The *geodetic closure* $I_G[S]$ is the union of intervals between all pairs of vertices from S, that is, $I_G[S] = \cup\{I_G[u, v] : u, v \in S\}$. A *geodetic set* of G is a set S with $I_G[S] = V(G)$. The *geodetic number* $g(G)$ of a graph G is the minimum cardinality of a geodetic set.

The reseachers find it interesting to study the geodetic set S of G in which the subgraph induced by the set S, denoted by $\langle S \rangle$, is connected and contains a path P, where $V(P) = S$. Such sets are called path-induced geodetic sets. The researchers believe that the concept on path-induced geodetic numbers of graphs can be applied in travel time saving, facility location, goods distribution, and other things in which this concept will be of great help.

2 Preliminary Concepts and Results

Definition 2.1 [2] The degree of a vertex v in G is the number of edges incident with v and denoted by $deg_G(v)$. A vertex is called an *end-vertex* if its degree is 1.

Definition 2.2 [2] The *removal of a vertex* v from a graph G results in the subgraph $G - v$ consisting of all vertices in G except v and all edges not incident with v.

Definition 2.3 [2] A vertex x of a graph G is called a cut-vertex if the removal of x increases the number of components of a graph G. We will use $w(G)$ to describe the number of components a graph G has.

Definition 2.4 [2] In a graph G, the *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set consisting of all vertices u which are adjacent to v, that is, $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. A vertex v in G is an *extreme vertex* if the neighborhood $N_G(v)$ of v induces a complete subgraph of G.

Definition 2.5[2] A nontrivial connected graph without cut-vertices is called *non-separable* graph. Otherwise, such graphs are *separable*.

Definition 2.6 [2] Let G be a nontrivial connected graph. A *block B* of G is a subgraph of G that is itself nonseparable and which is maximal with respect to this property.

Definition 2.7 [2] A *spanning subgraph* is a subgraph containing all the vertices of G. If that subgraph is a path, then it is called a *spanning path* of G.

Definition 2.8 Given a connected graph G and $S \subseteq V(G)$, the set S is called a *path-induced geodetic set* of G denoted by *pig-set*, if it satisfies the following properties:

1. $I_G[S] = V(G)$.
2. $\langle S \rangle$ is connected.
3. $\langle S \rangle$ contains a path P, where $V(P) = S$.

The minimum cardinality of a path-induced geodetic set is called *path-induced geodetic number* of G, denoted by $pign(G)$. A path-induced geodetic set of smallest cardinality is called a *path-induced geodetic basis* of G.

Example 2.9 Consider the graph G in Figure 1 below.

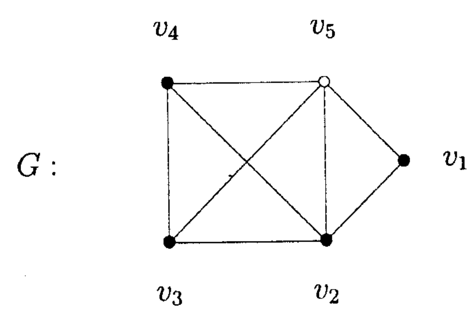

Figure 1: A graph G *with* $pign(G) = 4$.

Let $S_1 = \{v_1, v_3, v_4\}$. Now, we have

$$\begin{aligned} I_G[S_1] &= I_G[\{v_1, v_3, v_4\}] \\ &= I_G[v_1, v_3] \cup I_G[v_1, v_4] \cup I_G[v_3, v_4] \\ &= \{v_1, v_2, v_5, v_3\} \cup \{v_1, v_2, v_5, v_4\} \cup \{v_3, v_4\} \\ &= \{v_1, v_2, v_3, v_4, v_5\} \\ &= V(G). \end{aligned}$$

But $\langle S_1 \rangle$ is not connected. Hence, it is not a path-induced geodetic set. So we will add another vertex. Let $S = \{v_1, v_2, v_3, v_4\}$. Note that $I_G[S] = V(G)$. Moreover, $<S>$ is connected. Furthermore, $<S>$ contains a path P, where $V(P) = S$. P can be the path $[v_1, v_2, v_3, v_4]$ or $[v_1, v_2, v_{34}, v_3]$. Consequently, S is a *pig-set* of G and by definition, $pign(G) \leq |S| = 4$. Clearly, $pign(G) \neq 1$. Also, there is no set S of cardinality less than 4 for which S is a *pig-set*. Hence, $S = \{v_1, v_2, v_3, v_4\}$ is a path-induced geodetic basis of G. That is, $pign(G) = 4$.

Proposition 2.10[3] *Every extreme vertex is an end vertex of every geodesic containing it.*

Theorem 2.11[3] *For integers* $m, n \geq 2$, $g(K_{m,n}) = \min\{m, n, 4\}$.

Theorem 2.12[3] *For any integer* $n \geq 2$, $g(P_n) = 2$.

3 Path-Induced Geodetic Numbers of Some Graphs :

First, we have to remark that not all connected graphs have path-induced geodetic set. To illustrate this, let us have the following example.

Example 3.1 Consider the star S_4 in Figure 2. Observe that the only geodetic sets of S_4 are the sets $S = \{v_1, v_2, v_3, v_4\}$ and $S^* = \{v_1, v_2, v_3, v_4, v\}$. But, $\langle S \rangle$ is a totally disconnected graph, and so, not a path-induced geodetic set. On the other hand, $\langle S^* \rangle$ is S_4 itself and hence, $\langle S^* \rangle$ is connected. But, $\langle S^* \rangle$ does not contain a path P where $V(P) = S^*$. Thus, S^* is also not a path-induced geodetic set. That is, S_4 does not have a *pig-set.*

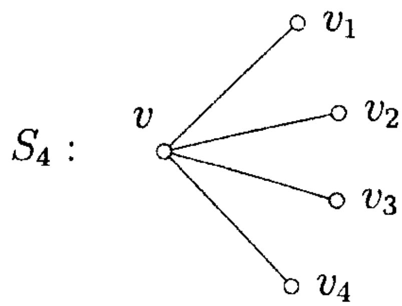

Figure 2: A graph without a *pig*-set

In general, the graph S_n does not have a path-induced geodetic set, for all $n \geq 3$.

To this extent, we will give conditions for a connected simple graph G to have a path-induced geodetic set. Let us have first these following results.

Theorem 3.2 Let G be a connected graph with cut-vertices. If $S \subseteq V(G)$ is a path-induced geodetic basis of G and x is a cut-vertex of G, then every component of $G - x$ contains a vertex in S.

Proof : Let x be a cut-vertex of G and S be a path-induced geodetic basis of G. Let $C_1, C_2, \ldots, C_k$ be the components of $G - x$. It remains to show that each C_i contains a vertex in S. Suppose that there exists a component, say C_1, of $G - x$ that does not contain a vertex in S. Further, let $V(C_1) = \{r_1, r_2, \ldots, r_t\}$, for some integer t. Since S is a path-induced geodetic basis of G, there exist $u, v \in S$ such that $r_i, 1 \leq i \leq t$, lies in some u-v geodesic. But this is impossible since the walk we can make starting from vertex u and ends with vertex v traversing vertex r_i is given by $[u = u_0, u_1, \ldots, x, \ldots, r_i, \ldots, x, \ldots, u_l = v]$, which is not a u-v geodesic. This is a contradiction. Hence, C_1 must contain a vertex in S. ■

Lemma 3.3 Every cut-vertex of a connected graph G belongs to every path- induced geodetic basis of G.

Proof : Let G be a connected graph and S be a path-induced geodetic basis of G. Suppose that x is a cut-vertex of G and let $C_1, C_2, \ldots, C_k$ be the components of $G - x$. By Theorem 3.2, S contains at least one vertex from each C_i, $i = 1, 2, \ldots, k$. Since the subgraph $\langle S \rangle$ of G is connected, it follows that $x \in S$. ■

The following three results are useful in determining whether a graph G has a path-induced geodetic set or none.

Theorem 3.4 Let G be a connected nontrivial graph of order n. If G contains a spanning path, then G has a path-induced geodetic set.

Proof : Let G be a connected nontrivial graph of order n and let G contains a spanning path. Take $S = V(G)$. Then, $I_G[S] = V(G)$. Moreover, $\langle S \rangle$ is the graph G itself and hence, connected. By assumption, G contains a spanning path P, that is, $V(P) = V(G) = S$. Therefore, S is a path-induced geodetic set of G. ■

In view of Theorem 3.4, if a graph G contains a spanning path, then we can automatically proceed in finding the path-induced geodetic number of G.

Theorem 3.5 Let G be a connected graph with cut-vertices. If G has a path-induced geodetic set, then $w(G - x) = 2$ for every cut-vertex x of G.

Proof : Let G be a connected graph and let S be a path-induced geodetic basis of G. Let $x \in V(G)$ be a cut-vertex. Suppose on the contrary that $w(G - x) \geq 3$ and let $C_1, C_2, \ldots, C_k, k \geq 3$, be the components of $G - x$. By Lemma 3.3 and Theorem 3.2, $x \in S$ and S contains a vertex of C_i for all i. Hence, $S = \{S_1 \cup S_2 \cup \ldots \cup S_k \cup x : S_i \subseteq C_i, k \geq 3\}$. By definition, $\langle S \rangle$ contains a path P, where $V(P) = S$. But this is impossible to happen since the walk we can make traversing all vertices of $\langle S \rangle$ is given by $[v_{11}, v_{12}, \ldots, v_{1r}, x, v_{21}, v_{22}, \ldots, v_{2s}, x, \ldots, v_{k1}, v_{k2}, \ldots, v_{kt}]$, $v_{ij} \in S_i$, which is not a path. This is a contradiction. Hence, $w(G - x) = 2$. ■

The contrapositive of Theorem 3.5 says that if there exists a cut-vertex x of G with $w(G - x) \geq 3$, then G has no path-induced geodetic set.

Figure 3 shows an illustration of the situation described in Theorem 3.5. $w(G - v_2) = 3$ and therefore does not allow G to have a path-induced geodetic set.

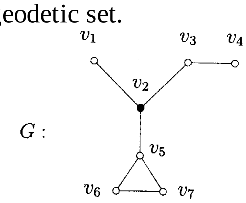

Figure 3: A graph without a pig-set

Theorem 3.6 Let G be a connected graph with cut-vertices. If G has a path-induced geodetic set, then each block of G contains at most 2 *cut-vetices.*

Proof : Let G be a connected graph with cut-vertices and let S be a path- induced geodetic basis of G. Suppose there is a block B of G with cut-vertices v_1, v_2, v_3. By Theorem 3.5, $w(G-v_i) = 2$, $i = 1, 2, 3$. Observe that there is a component of $G-v_i$, say B_i, in which $V(B_i) \cap V(B) = \theta$. By Theorem 3.2, each B_i contains a vertex in S. By assumption, there is a path P that contains all the vertices in S. But to get from any vertex in $V(B_i)$ to a vertex $x \in V(B_j)$, it needs to pass through v_i. To get from v_i to x, the path must pass through v_j. Using the same argument as in the proof of Theorem 3.5, we see that P needs to visit one of the cut-vertices v_1, v_2, v_3 at least twice, which is a contradiction to P being a path. ■

The contrapositive of Theorem 3.6 says that if there exists a block of G with three or more cut-vertices, then G has no path-induced geodetic set.

Figure 4 shows an illustration of the situation described in Theorem 3.6. Block B has three cut-vertices v_1, v_2, v_3 and therefore does not allow G to have a path-induced geodetic set.

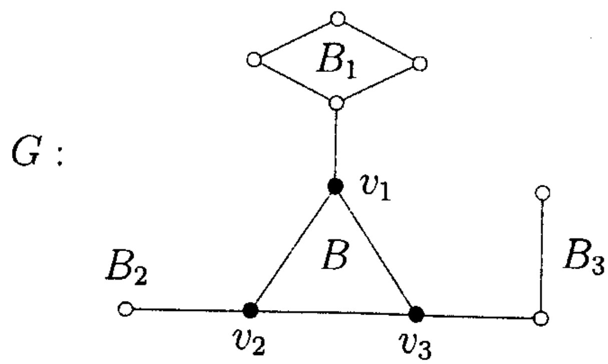

Figure 4: A graph without a I-set

From this point onwards, the graph G considered has path-induced geodetic set.

Remark 3.7 Since every path-induced geodetic set S is a geodetic set of G, we have $g(G) \le pign(G)$.

Now, since any geodetic set needs at least two vertices while the maximum number of vertices a path-induced geodetic set can have is the order of G, together with Remark 3.7, we have the following remark.

Remark 3.8 Let G be a connected nontrivial graph of order n. Then

$$2 \le g(G) \le pign(G) \le n.$$

Theorem 3.9 Let G be a connected graph and S be a path-induced geodetic basis of G. Then every extreme vertex of G is contained in S.

Proof: Let G be a graph and S be a path-induced geodetic basis of G. Suppose on the contrary that there exists an extreme vertex u of G which is not contained in S. However, by definition of S, $I_G[S] = V(G)$, so there exist $x, y \in S$ such that $u \in I_G[x, y]$. This implies that u lies in the x-y geodesic and by Proposition 2.10, u must be an end-vertex, that is, $u = x$ or $u = y$. But this is impossible to happen since x and y are in S, a contradiction. Therefore, $u \in S$. ■

Remark 3.10 Every vertex of a complete graph K_n is an extreme vertex.

Theorem 3.11 For any natural number n, pign$(K_n) = n$.

Proof : Let V $(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let S be a path-induced geodetic basis of K_n . By Remark 3.10 and Theorem 3.9, $v_i \in S$ $\forall i = 1, 2, \ldots, n$. That is, $S = \{v_1, v_2, \ldots, v_n\}$. Therefore, $pign(K_n) = n$. ■

The following theorem is one of the necessary conditions of a graph G of order n to have $pign(G) = n$.

Theorem 3.12 Let G be a connected graph with $|V(G)| = n$. *If every vertex of G is either an extreme vertex or a cut-vertex of G, then* $pign(G) = n$.

Proof : Let G be a graph with $|V(G)| = n$ and S be a path-induced geodetic basis of G. Suppose $v \in V(G)$. If v is an extreme vertex, then by Theorem 3.9, $v \in S$. Also, if v is a cut-vertex, then by Lemma 3.3, $v \in S$. Hence, in either case, $v \in S$. Therefore, $S = V(G)$ and thus, $pign(G) = n$. ■

Remark 3.13 Every end-vertex in a graph G is an extreme vertex.

Theorem 3.14 Let $G = P_n$. *Then* $pign(G) = n$, *for all* $n \geq 2$.

Proof : Let $G = P_n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$, as shown in Figure 5.

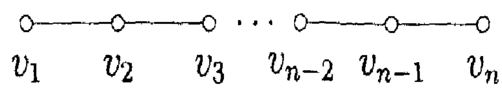

Figure 5: A path P_n of order n, $n \geq 2$

Let S be a path-induced geodetic basis of P_n. Note that v_1 and v_n are end-vertices and by Remark 3.13, v_1, v_2 are extreme vertices of P_n. Also, vertices $v_2, \ldots, v_{n-1}$ are cut-vertices since $w(P_n - v_i) = 2$ for each $i = 2, 3, \ldots, n - 1$. Hence, every vertex of P_n is either an extreme vertex or a cut-vertex. By Theorem 3.12, $pign(P_n) = n$. ■

The following result is immediate:

Corollary 3.15 If $n = 2$, *then* $pign(P_n) = g(P_n)$.

The next theorem characterizes those connected nontrivial graphs G for which the path-induced geodetic number is 2.

Theorem 3.16 Let G be a connected nontrivial graph. Then, $pign(G) = 2$ *if and only if* $G = P_2$.

Proof : Let $S = \{u, v\}$ be a path-induced geodetic basis of G. Suppose $G \neq P_2$. Then there exists $x \in V(G)$ such that $x \neq u$ and $x \neq v$. But, $x \notin I_G[u, v]$ since $d_S(u, v) = d_G(u, v) = 1$. Thus, S is not a path-induced geodetic basis of G, a contradiction. Hence, $G = P_2$. Conversely, If $G = P_2$, then $pign(G) = 2$ by Theorem 3.14. ■

The next theorem gives the formula on how to get $pign(C_n)$ for $n \geq 3$.

Theorem 3.17 Let $G = C_n$. *Then for* $n \geq 3$,

$$pign(C_n) = \begin{cases} \frac{n+1}{2} + 1, & if\ \ n\ is\ odd \\ \frac{n}{2} + 1, & if\ \ n\ is\ cvcn. \end{cases}$$

Proof : Let $G = C_n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider the following cases for the order n of graph G.

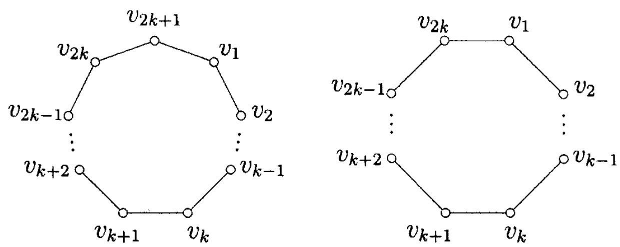

Figure 6: Cycle Cn of order n

Case 1: When n is odd, say $n = 2k + 1$, for some integer k.
For a cycle C_n of order $n = 2k + 1$, it is the union of two paths $[v_{2k+1}, v_1, v_2, \ldots, v_{k+1}]$ and $[v_{k+1}, v_{k+2}, \ldots, v_{2k+1}]$ of lengths $k+1$ and k, respectively. Let $S = \{v_{2k+1}, v_1, v_2, \ldots, v_{k+1}\}$. Observe that $I_G[S]$ contains the vertices on $[v_{k+1}, v_{k+2}, \ldots, v_{2k+1}]$ where $\{v_{k+1}, v_{k+2}, \ldots, v_{2k+1}\} = I_G[v_{k+1}, v_{2k+1}]$. Thus, $I_G[S] = V(G)$. Moreover, $\langle S\rangle$ is connected and contains a path P where $V(P) = S$. In fact, P is $\langle S\rangle$ itself. Thus, S is a *pig*-set. Moreover, there is no *pig*-set of cardinality less than that of S. Therefore,

$$pign(G) = |S| = |\{v_1, v_2, \ldots, v_{k+1}\} \cup \{v_n = v_{2k+1}\}| = (k+1) + 1 = \frac{n+1}{2} + 1.$$

Case 2: When n is even, say $n = 2k$, for some integer k.
Observe that paths $[v_1, v_2, \ldots, v_{k+1}]$ and $[v_{k+1}, v_{k+2}, \ldots, v_1]$ are of the same length. Hence, $I_G[v_1, v_{k+1}] = V(G)$. That is, the minimum number of vertices for a geodetic set S where $\langle S\rangle$ is connected and contains a path P with $V(P) = S$ is attained if $S = \{v_1, v_2, \ldots, v_{k+1}\}$ or $S = \{v_{k+1}, v_{k+2}, \ldots, v_{2k}, v_1\}$. Therefore, $pign(G)$

$$= |S| = |\{v_1, v_2, \ldots, v_{k+1}\}| = k+1 = \frac{n}{2}1.$$ ■

Lemma 3.18 For all integers $m, n > 2$, let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ *be partite sets of $K_{m,n}$. A subset S of $V(K_{m,n})$ is a path-induced geodetic set of $K_{m,n}$ if and only if S is any of the following:*

1. $S = A \cup B$, *where* $A \subseteq U, B \subseteq W$ *with* $|A| = |B| = r$, $2 \le r \le \min\{m, n\}$;
2. $S = A \cup B$, *where* $A \subseteq U, B \subseteq W$ *with* $|A| = s$, $|B| = s + 1$, $2 \le s \le \min\{m, n-1\}$;
3. $S = A \cup B$, *where* $A \subseteq U, B \subseteq W$ *with* $|A| = c + 1$, $|B| = c$, $2 \le c \le \min\{m-1, n\}$.

Proof: ($\Leftarrow$) *Let* $G = K_{m,n}$ and let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be partite sets of G, where $m, n > 2$.

(1) Let $S = A \cup B$, where $A \subseteq U$, $B \subseteq V$ with $|A| = |B| = r$, $2 \le r \le \min\{m, n\}$. Without loss of generality, assume that $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{w_1, w_2, \ldots, w_r\}$. Note that $I_G[u_i, u_j] = W$ for any $u_i, u_j \in A$, $i \ne j$, since $[u_i, w, u_j]$ is a u_i-u_j geodesic in G for all $w \in W$. Also, $I_G[w_k, w_l] = U$ for any $w_k, w_l \in W$, $k \ne l$, since $[w_k, u, w_l]$ is a w_k - w_l geodesic in G for all $u \in U$. Thus, $I_G[S] = V(G)$. Since every vertex in A is adjacent to every vertex in B, $\langle S\rangle$ is connected. Moreover, $\langle S\rangle$ contains a path P where $V(P) = S$. The path P can be the path $[u_1, w_1, u_2, w_2, \ldots, u_r, w_r]$. Therefore, S is a path-induced geodetic set of G.

(2) Let $S = A \cup B$, where $A \subseteq U$, $B \subseteq W$ with $|A| = s$, $|B| = s + 1$, $2 \leq s \leq \min\{m, n - 1\}$. Without loss of generality, assume that $A = \{u_1, u_2, \ldots, u_s\}$ and $B = \{w_1, w_2, \ldots, w_s, w_{s+1}\}$. From the proof of (1), $I_G[u_i, u_j] = W$ for any two distinct vertices $u_i, u_j \in A$ and $I_G[w_k, w_l] = U$ for any two distinct vertices $w_k, w_l \in W$. Hence, it follows that $I_G[S] = V(G)$. Also, $\langle S \rangle$ is connected since every vertex in A is adjacent to every vertex in B. Moreover, $\langle S \rangle$ contains a path P where $V(P) = S$. The path P can be the path $[w_1, u_1, w_2, u_2, \ldots, w_s, u_s, w_{s+1}]$. Therefore, S is a path-induced geodetic set of G.

(3) Let $S = A \cup B$, where $A \subseteq U$, $B \subseteq W$ with $|A| = c + 1$, $|B| = c$, $2 \leq c \leq \min\{m - 1, n\}$. Without loss of generality, assume that $A = \{u_1, u_2, \ldots, u_c, u_{c+1}\}$ and $B = \{w_1, w_2, \ldots, w_c\}$. Following similar argument at the proof of (2), $I_G[S] = V(G)$, $\langle S \rangle$ is connected and $\langle S \rangle$ contains a path P where $V(P) = S$. The path P can be the path $[u_1, w_1, u_2, w_2, \ldots, u_c, w_c, u_{c+1}]$. Consequently, S is a path-induced geodetic set of G.

($\Rightarrow$) Let S be a path-induced geodetic set of G. By definition, $\langle S \rangle$ is connected. Hence, S must contain a vertex in U and a vertex in W, that is, $S = A \cup B$, where $A \leq U$, $B \leq W$. Also, $\langle S \rangle$ contains a path P where $V(P) = S$. The path P is an alternating sequence of vertices from sets A and B, otherwise, P will be disconnected. If P begins with a vertex in A and ends with a vertex in B, or vice versa, then $|A| = |B|$ while if P begins with a vertex in B [resp. A] and ends also with a vertex in B [resp. A], then $|B| = |A| + 1$ [resp. $|A| = |B| + 1$].

(i) For the case where $|A| = |B| = r$:

Claim: $r \geq 2$

Suppose on the contrary that $r < 2$, that is, $r = 1$. Let $A = \{u_i\}$ and $B = \{w_j\}$. Then, $I_G[S] = I_G[u_i, w_j] = \{u_i, w_j\} \neq V(G)$. Hence, S is not a path-induced geodetic set, a contradiction. Therefore, $r \geq 2$.

Moreover, the maximum number of vertices a subset of U can have is m while the maximum number of vertices a subset of W can have is n. Hence, it follows that $|A| = |B| = r \leq \min\{m, n\}$. Therefore, $2 \leq r \leq \min\{m, n\}$, which gives us the *pig*-set S in (1).

(ii) For the case where $|B| = |A| + 1$:

Let $|A| = s$. Then $|B| = s + 1$.

Claim: $s \geq 2$

Suppose on the contrary that $s < 2$, that is, $s = 1$. Let $A = \{u_i\}$ and $B = \{w_k, w_l\}$. Then,

$$\begin{aligned} I_{K_{m,n}}[S] &= I_{K_{m,n}}[\{u_i, w_k, w_l\}] \\ &= I_{K_{m,n}}[u_i, w_k] \cup I_{K_{m,n}}[u_i, w_l] \cup I_{K_{m,n}}[w_k, w_l] \\ &= \{u_i, w_k\} \cup \{u_i, w_l\} \cup U \\ &= U \cup \{w_k, w_l\} \neq V(K_{m,n}), \text{ since } m, n > 2. \end{aligned}$$

Hence, S is not a path-induced geodetic set, a contradiction. Therefore, $s \geq 2$. Moreover, the maximum number of vertices a subset of U can have for this case is $s = m$ while the maximum number of vertices a subset of W can have is $s = n - 1$, so that $s + 1 = n$. Hence, it follows that $s \leq \min\{m, n - 1\}$. Therefore, $2 \leq s \leq \min\{m, n - 1\}$, which gives us the *pig*-set S in (2).

(iii) For the case where $|A| = |B| + 1$, similar argument from the latter case is applied. Then, we will get the range for $c = |B|$, that is, $2 \leq c \leq \min\{m-1, n\}$, which gives us the *pig*-set S in (3). ■

Theorem 3.19 For all integers $m, n > 2$,

$$pign(K_{m,n}) = 4.$$

Proof: Let U and W be partite sets of $K_{m,n}$, where $|U| = m$, $|W| = n$, $m, n > 2$. By Lemma 3.18, the only path-induced geodetic sets of $K_{m,n}$ are (1) $S = A \cup B$, where $A \subseteq U, B \subseteq W$ with $|A| = |B| = r$, $2 \leq r \leq \min\{m, n\}$; (2) $S = A \cup B$, where $A \subseteq U$, $B \subseteq W$ with $|A| = s$, $|B| = s + 1$, $2 \leq s \leq \min\{m, n - 1\}$ and (3) $S = A \cup B$, where $A \subseteq U$, $B \subseteq W$ with $|A| = c + 1$, $|B| = c$, $2 \leq c \leq \min\{m - 1, n\}$. Note that the cardinality of the *pig*-set S in (1) is given by $4 \leq |S| \leq 2\cdot \min\{m, n\}$ while the *pig*-set S in (2) is given by $5 \leq |S| \leq 2 \cdot \min\{m, n - 1\} + 1$. Also, the cardinality of the *pig*-set S in (3) is given by $5 \leq |S| \leq 2 \cdot \min\{m - 1, n\} + 1$. Hence, $pign(K_{m,n}) = \min\{|S| : S \text{ is a } pig\text{-set of } K_{m,n}\} = 4$. ■

Corollary 3.20 Let $m, n > 2$. *Then,* $pign(K_{m,n}) = g(K_{m,n})$ if and only if $\min\{m, n\} \geq 4$.

Proof : Let $pign(K_{m,n}) = g(K_{m,n})$. Suppose on the contrary that $\min\{m, n\} \leq 3$, and since $m, n > 2$, we have $\min\{m, n\} = 3$. By Theorems 2.11 and 3.19, $g(K_{m,n}) = \min\{m, n, 4\}$ and $pign(K_{m,n}) = 4$, respectively. Since $\min\{m, n\} = 3$, it follows that $g(K_{m,n}) = 3$ while $pign(K_{m,n}) = 4$ for all $m, n > 2$. This implies that $pign(K_{m,n}) \neq g(K_{m,n})$, a contradiction. Therefore, $\min\{m, n\} \geq 4$.

Conversely, suppose that $\min\{m, n\} \geq 4$. By Theorems 2.11 and 3.19, $g(K_{m,n}) = \min\{m, n, 4\}$ and $pign(K_{m,n}) = 4$, respectively. Since $\min\{m, n\} \geq 4$, it follows that $g(K_{m,n}) = 4 = pign(K_{m,n})$. ■

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