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# Quotient Hyper BCK-algebra and its Zero Divisor Graph

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**Abstract**

This study investigates some properties of regular congruence relations on a hyper BCK-algebra. Furthermore, the concept of zero divisor graph of quotient hyper BCK-algebra is introduced and some of its properties are investigated.

*Key words:* zero divisor graph, hyper BCK-algebra, quotient hyper BCK-algebra

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## 1 Introduction

I. Beck[2] introduced the notion of zero divisor graph of a commutative ring in 1988. He associated a simple graph $G(R)$ to any commutative ring $R$ whose vertices are the elements of $R$, such that two vertices $x, y \in R$ are adjacent if and only if $xy = 0$. Since then, several authors studied the zero divisor graph of commutative rings and extended the concept to zero divisor graph of commutative semigroups. In[1], D.F. Anderson and E. F. Lewis introduced a unifying concept of zero divisor graph over a commutative ring $R$ based on a multiplicative congruence relation ~ on $R$. Then $R/{\sim}=\{[x]_{\sim}|x\in R\}$, the set of ~ -congruence classes of $R$, is a commutative monoid under the induced multiplication $[x]_{\sim}[y]_{\sim}=[xy]_{\sim}$ with identity element $[1]_{\sim}$ and zero element $[0]_{\sim}$. The ~-congruence-based zero divisor graph $G(R/{\sim})$ with respect to ~ is the graph whose vertices are the nonzero zero divisors of $R/{\sim}$ and with distinct vertices $[x]_{\sim}$ and $[y]_{\sim}$ are adjacent if and only if $[x]_{\sim}[y]_{\sim} = [xy]_{\sim} = [0]_{\sim}$.

In[5], Y. B. Jun and K. J. Lee introduced the concept of associated graph of BCK-algebra and verified some properties of this graph. In[8], O. Zahiri and R. A. Borzooei extended the work of Jun and Lee and investigated the relation between some operations on graphs and some operations on BCK-algebras. This paper will investigate some properties of quotient hyper BCK-algebra and its zero divisor graph.

## 2 Preliminaries

A hyperoperation on a non-empty set $H$ is a map from $H \times H$ into $P^*(H) = P(H) \setminus \{\varnothing\}$. Let "*" be a hyperoperation on $H$ and $(x, y) \in H \times H$. For two nonempty subsets $A$ and $B$ of $H$, $A * B = \bigcup_{a\in A, b\in B} a * b$. We shall use $x * y$ instead of $x*\{y\}$, $\{x\}*y$ or $\{x\}*\{y\}$. When $A$ is a nonempty subset of $H$ and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a\in A} a * x$ and $x * A = \bigcup_{a\in A} x * a$.

A *hyper BCK-algebra* is a nonempty set $H$ endowed with a hyperoperation "*" and a constant $0$ satisfying the following axioms: for all $x, y, z \in H$, (H1) $(x*z)*(y*z) \ll x*y$; (H2) $(x*y)*z = (x*z)*y$; (H3) $x * H \ll x$; and (H4) $x \ll y$ and $y \ll x$ imply $x = y$, where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a * b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x * y$. In such case, we call " $\ll$ " the *hyper order* in $H$. The properties of hyper BCK-algebra $(H,*,0)$ are discussed in[7].

Throughout the paper, denote a hyper BCK-algebra $(H,*,0)$ simply by $H$ if no confusion arises, unless otherwise stated.

*Definition 2.1*[6,7] Let $I$ be a nonempty subset of a hyper BCK-algebra $H$. Then $I$ is said to be a *hyper BCK-ideal* of $H$ if $0 \in I$; and $x * y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$. If $(x * y) \cap I \neq \varnothing$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, then $I$ is said to be a *strong hyper BCK-ideal* of $H$.

*Theorem 2.2*[7] *In any hyper BCK-algebra* $H$, $0 \ll x$, $x \ll x$ *and* $x * 0 = \{x\}$ *for all* $x \in H$.

*Definition 2.3*[3] Let $H$ be a hyper BCK-algebra and $\Theta$ be an equivalence relation on $H$ and $A, B \subseteq H$. Then,

(i) $A\Theta B$ means that, there exists $a \in A$ and $b \in B$ such that $a\Theta b$,

(ii) $A\overline{\Theta}B$ means that, for all $a \in A$ there exists $b \in B$ such that $a\Theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\Theta b$,

(iii) $\Theta$ is called a *congruence relation* on $H$, if $x\Theta y$ and $x'\Theta y'$ then $x * x'\overline{\Theta} y * y'$, for all $x, y, x', y' \in H$,

(iv) $\Theta$ is called a *regular relation* on $H$, if $x * y\Theta\{0\}$ and $y * x\Theta\{0\}$, then $x\Theta y$ for all $x, y \in H$.

*Theorem 2.4* [3] *Let* $\Theta$ *be a regular congruence relation on* $H$ *and* $H/\Theta = \{[x]_\Theta : x \in H\}$. *Then* $H/\Theta$ *with hyperoperation "*" and hyperorder "*$\ll$*" which is defined as follows,*

$$[x]_\Theta * [y]_\Theta = \{[z]_\Theta : z \in x * y\}, [x]_\Theta \ll [y]_\Theta \Leftrightarrow [0]_\Theta \in [x]_\Theta * [y]_\Theta$$

*is a hyper BCK-algebra which is called quotient hyper BCK-algebra.*

*Lemma 2.5*[3] *Let* $\Theta$ *be a regular congruence relation on* $H$. *Then* $[0]_\Theta$ *is a strong hyper BCK-ideal of* $H$.

*Definition 2.6* [3] Let $H$ and $H'$ be two hyper BCK-algebras and $f : H \to H'$ be a map. Then $f$ is said to be a *homomorphism* of hyper BCK-algebras if $f(x * y) = f(x) * f(y)$, for all $x, y \in H$. If $f$ is one-to-one (onto), we say that $f$ is a *monomorphism* (*epimorphism*). In addition, if $f$ is both one-to-one and onto, we say that $f$ is an *isomorphism*. If $f : H \to H'$ is an isomorphism, then we say that $H$ and $H'$ are isomorphic and we write $H \cong H'$.

A *graph* $G$ is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex set* of $G$ and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply $uv$) of distinct elements from $V(G)$ called the *edge set* of $G$. A graph $K = (V(K), E(K))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(K) \subseteq V(G)$ and $E(K) \subseteq E(G)$. Two vertices $u, v$ of a graph $G$ are *adjacent*, or *neighbors*, if $uv$ is an edge of $G$. Moreover, an edge $uv$ of $G$ is *incident* to two vertices $u, v$ of $G$. A graph $G$ is called an *empty graph* if $E(G) = \varnothing$. Let $G$ and $K$ be graphs and let $f : V(G) \to V(K)$ be a function. Then $f$ is a *graph homomorphism* if $xy \in E(G)$, then $f(x)f(y) \in E(K)$. Two graphs $G$ and $K$ are *isomorphic* (written as $G \cong K$) if there exists a one-to-one correspondence between the vertex sets which preserves adjacency,[4].

## 3 On Regular Congruence Relation of Quotient Hyper BCK-algebra

*Lemma 3.1 Let* $\Theta$ *be a regular congruence relation on* $H$ *which is a hyper BCK- algebra and* $H/\Theta = \{[x]_\Theta : x \in H\}$. *Then* $[x]_\Theta \ll [y]_\Theta$ *if and only if* $x \ll y$ *for all* $x, y \in H$.

*Proof*: Suppose $[x]_\Theta \ll [y]_\Theta$. Then $[0]_\Theta \in [x]_\Theta * [y]_\Theta$. By Theorem 2.4, $0 \in x * y$. Thus, $x \ll y$. Conversely, suppose $x \ll y$. Then $0 \in x * y$. By Theorem 2.4, $[0]_\Theta \in [x]_\Theta * [y]_\Theta$. Thus, $[x]_\Theta \ll [y]_\Theta$. ■

*Theorem 3.2 Let* $H$ *be a hyper BCK-algebra. Then* $\Theta_H = \{(x, x) \mid x \in H\}$ *and* $H \times H$ *are regular congruence relations.*

*Proof*: Clearly, $\Theta_H$ is an equivalence relation on $H$. We will show that $\Theta_H$ is a regular congruence relation. Let $x\Theta_H y$ and $x'\Theta_H y'$ where $x, y, x', y' \in H$. Then $x = y$ and $x' = y'$. Thus, $x * x' = y * y'$. Hence, $t \in x * x'$ if and only if $t \in y * y'$. Since $t\Theta_H t$, it follows that $x * x'\overline{\Theta}_H y * y'$. This implies that $\Theta_H$ is a congruence relation. Let $x, y \in H$ such that $x * y\Theta_H\{0\}$ and $y * x\Theta_H\{0\}$. Then there exist $s \in x * y$ and $t \in y * x$ such that $s\Theta_H 0$ and $t\Theta_H 0$. Thus, $s = 0 = t$ which means $0 \in x * y$ and $0 \in y * x$. Hence, $x \ll y$ and $y \ll x$. This implies that $x = y$ and so $x\Theta_H y$. Therefore, $\Theta_H$ is a regular congruence relation. Clearly, $H \times H$ is a regular congruence relation. ■

*Definition 3.3* Let $Con_R(H)$ be the set of all regular congruence relations on $H$. Then $Con_R(H)$ is partially ordered by inclusion, that is, for $\Theta_1, \Theta_2 \in Con_R(H)$,

$$\begin{aligned}\Theta_1 \leq \Theta_2 &\Leftrightarrow \Theta_1 \subseteq \Theta_2 \text{ as subsets of } H \times H\\ &\Leftrightarrow x\Theta_1 y \Rightarrow x\Theta_2 y \text{ for } x, y \in H\\ &\Leftrightarrow [x]_{\Theta_1} \subseteq [x]_{\Theta_2} \text{ for every } x \in H.\end{aligned}$$

Moreover, $Con_R(H)$ has a least element $\Theta_H = \{(x, x) : x \in H\}$ and a greatest element $H \times H$.

*Lemma 3.4 Let $H$ be a hyper BCK-algebra and let $\Theta_1, \Theta_2 \in Con_R(H)$. Then $[x]_{\Theta_1} \ll [y]_{\Theta_1}$ if and only if $[x]_{\Theta_2} \ll [y]_{\Theta_2}$.*

*Proof*: Suppose $[x]_{\Theta_1} \ll [y]_{\Theta_1}$. By Lemma 3.1, $x \ll y$ which implies that $[x]_{\Theta_2} \ll [y]_{\Theta_2}$. The proof of converse is similar. ■

*Definition 3.5* For a strong hyper BCK-ideal $I$ of $H$, let $Con_R(H)_I = \{\Theta \in Con_R(H) \mid [0]_\Theta = I\}$.

*Proposition 3.6 Let $H$ be a hyper BCK-algebra. Then*

$$Con_R(H) = \bigcup\{Con_R(H)_I \mid I \text{ is a strong hyper BCK-ideal of } H\}$$

*Proof*: Let $C = \bigcup\{Con_R(H)_I \mid I \text{ is a strong hyper BCK-ideal of } H\}$ and $\Theta \in Con_R(H)$. By Lemma 2.5, $[0]_\Theta = I_0$ is a strong hyper BCK-ideal of $H$. Thus, $\Theta \in Con_R(H)_{I_0}$. Hence, $\Theta \in C$. Let $\Theta \in C$. By Definition 3.5, $\Theta \in Con_R(H)$. ■

*Remark 3.7 Let $H$ be a hyper BCK-algebra and $\Theta$ be a regular congruence relation on $H$.*

(i) If $\Theta = \{(x,x) \mid x \in H\}$, then $[0]_\Theta = \{0\}$ and $\Theta \in Con_R(H)_{\{0\}}$.

(ii) $\Theta = H \times H$ *if and only if* $[0]_\Theta = H$.

## 4 On Zero Divisor Graph of Quotient Hyper BCK-algebra

Let $H$ be a hyper BCK-algebra, $\Theta \in Con_R(H)$ and $A \subseteq H/\Theta$. We will use the notation $L_{H/\Theta}(A)$ to denote the set

$$L_{H/\Theta}(A) := \{[x]_\Theta \in H/\Theta \mid [x]_\Theta \ll a, \forall a \in A\}.$$

*Definition 4.1* Let $H$ be a hyper BCK-algebra and $\Theta$ be a regular congruence relation on $H$. The zero divisor graph $\Gamma(H/\Theta)$ of $H/\Theta$ is the graph whose vertex set $V(\Gamma(H/\Theta)) = H/\Theta$ and the edge set $E(\Gamma(H/\Theta))$ satisfies the following condition: for every distinct $[x]_\Theta, [y]_\Theta \in H/\Theta$, $[x]_\Theta[y]_\Theta \in E(\Gamma(H/\Theta))$ if and only if $L_{H/\Theta}(\{[x]_\Theta, [y]_\Theta\}) = \{[0]_\Theta\}$.

*Lemma 4.2 Let $H$ be a hyper BCK-algebra, $\Theta_1, \Theta_2 \in Con_R(H)$ such that $\Theta_1 \leq \Theta_2$ and $[x]_{\Theta_2} \neq [y]_{\Theta_2}$. If $[x]_{\Theta_1}[y]_{\Theta_1} \in E(\Gamma(H/\Theta_1))$, then $[x]_{\Theta_2}[y]_{\Theta_2} \in E(\Gamma(H/\Theta_2))$.*

*Proof*: Suppose $[x]_{\Theta_1}[y]_{\Theta_1} \in E(\Gamma(H/\Theta_1))$. Then $L_{H/\Theta_1}(\{[x]_{\Theta_1}, [y]_{\Theta_1}\}) = \{[0]_{\Theta_1}\}$. Let $[w]_{\Theta_2} \in L_{H/\Theta_2}(\{[x]_{\Theta_2}, [y]_{\Theta_2}\})$. Then $[w]_{\Theta_2} \ll [x]_{\Theta_2}$ and $[w]_{\Theta_2} \ll [y]_{\Theta_2}$. By Lemma 3.4, $[w]_{\Theta_1} \ll [x]_{\Theta_1}$ and $[w]_{\Theta_1} \ll [y]_{\Theta_1}$. So, $[w]_{\Theta_1} \in L_{H/\Theta_1}(\{[x]_{\Theta_1}, [y]_{\Theta_1}\}) = \{[0]_{\Theta_1}\}$. Since $\Theta_1 \leq \Theta_2$, $[w]_{\Theta_1} = [0]_{\Theta_1} \subseteq [0]_{\Theta_2}$. Hence, $[0]_{\Theta_2} = [w]_{\Theta_2}$. Therefore, $L_{H/\Theta_2}(\{[x]_{\Theta_2}, [y]_{\Theta_2}\}) = \{[0]_{\Theta_2}\}$ and $[x]_{\Theta_2}[y]_{\Theta_2} \in E(\Gamma(H/\Theta_2))$. ■

*Theorem 4.3 Let $H$ be a hyper BCK-algebra and $\Theta$ is a regular congruence relation on $H$. Then $\Gamma(H/\Theta)$ is an empty graph if and only if $[0]_\Theta = H$.*

*Proof*: Let $\Gamma(H/\Theta)$ be an empty graph and $[0]_\Theta \neq H$. Then $[0]_\Theta \subset H$. Thus, $H/\Theta$ has more than one element. Since all the other element of $H/\Theta$ is adjacent to $[0]_\Theta$, $\Gamma(H/\Theta)$ is not an empty graph, a contradiction. Hence, $[0]_\Theta = H$. Conversely, suppose $[0]_\Theta = H$. Then $H/\Theta = \{[0]_\Theta\}$. Therefore, $\Gamma(H/\Theta)$ is an empty graph. ■

*Lemma 4.4 Let $H$ be a hyper BCK-algebra and let $\Theta$ be a regular congruence relation. Then $[x]_\Theta * [y]_\Theta = [x * y]_\Theta$ for all $x, y \in H$.*

*Proof*: Let $x, y, z \in H$. Then $[x]_\Theta * [y]_\Theta = \{[z]_\Theta \in H/\Theta \mid z \in x * y\} = \{w \in [z]_\Theta \mid w\Theta z, z \in x * y\}$ $= \{w \in H \mid w\Theta x * y\} = [x * y]_\Theta$. ■

*Lemma 4.5 Let $H$ be a hyper BCK-algebra, $A \subseteq H$ and let $\Theta_1, \Theta_2 \in Con_R(H)$ such that $\Theta_1 \leq \Theta_2$. If $f : H/\Theta_1 \to H/\Theta_2$ is a function given by $f([x]_{\Theta_1}) = [x]_{\Theta_2}$, then $f([x * y]_{\Theta_1}) = [x * y]_{\Theta_2}$ for all $x, y \in H$.*

*Proof*: By Lemma 4.4, $f([x * y]_{\Theta_1}) = f([x]_{\Theta_1} * [y]_{\Theta_1}) = f(\{[z]_{\Theta_1} \mid z \in x * y\}) = \{f([z]_{\Theta_1}) \mid z \in x * y\} = \{[z]_{\Theta_2} \mid z \in x * y\} = [x]_{\Theta_2} * [y]_{\Theta_2} = [x * y]_{\Theta_2}$. ■

*Theorem 4.6 Let $H$ be a hyper BCK-algebra and let $\Theta_1, \Theta_2 \in Con_R(H)$ such that $\Theta_1 \leq \Theta_2$. Then there exists a surjective homomorphism $f : H/\Theta_1 \to H/\Theta_2$ given by $f([x]_{\Theta_1}) = [x]_{\Theta_2}$.*

*Proof*: Let $[x]_{\Theta_1}, [y]_{\Theta_1} \in H/\Theta_1$ such that $[x]_{\Theta_1} = [y]_{\Theta_1}$. Since $\Theta_1 \leq \Theta_2$, $[x]_{\Theta_1} \subseteq [x]_{\Theta_2}$ and $[y]_{\Theta_1} \subseteq [y]_{\Theta_2}$. Thus, $[y]_{\Theta_1} \subseteq [x]_{\Theta_2}$ and $[x]_{\Theta_1} \subseteq [y]_{\Theta_2}$. Since $[x]_{\Theta_1} \subseteq [x]_{\Theta_2} \cap [y]_{\Theta_2}$, $[x]_{\Theta_2} = [y]_{\Theta_2}$. Hence, $f([x]_{\Theta_1}) = f([y]_{\Theta_1})$. Therefore, $f$ is well-defined. Let $[x]_{\Theta_1}, [y]_{\Theta_1} \in H/\Theta_1$. By Lemmas 4.4 and 4.5, $f([x]_{\Theta_1} * [y]_{\Theta_1}) = f([x * y]_{\Theta_1}) = [x * y]_{\Theta_2} = [x]_{\Theta_2} * [y]_{\Theta_2} = f([x]_{\Theta_1}) * f([y]_{\Theta_1})$. Thus, $f$ is a homomorphism. Let $[x]_{\Theta_2} \in H/\Theta_2$. Then $[x]_{\Theta_1} \in H/\Theta_1$ and $f([x]_{\Theta_1}) = [x]_{\Theta_2}$. Hence, $f$ is surjective. ■

*Corollary 4.7 Let $H$ be a hyper BCK-algebra and let $\Theta_H = \{(x, x) \mid x \in H\}$. Then there exists a surjective homomorphism $f : H/\Theta_H \to H/\Theta$ for all $\Theta \in Con_R(H)$ given by $f([x]_{\Theta_H}) = [x]_\Theta$.*

*Proof*: Since $\Theta_H$ is the least element of $Con_R(H)$, $\Theta_H \leq \Theta$. By Theorem 4.6, there exists a surjective homomorphism $f : H/\Theta_H \to H/\Theta$ for all $\Theta \in Con_R(H)$ given by $f([x]_{\Theta_H}) = [x]_\Theta$. ■

Consider a hyper BCK-algebra $H$ and let $\Theta_1, \Theta_2 \in Con_R(H)$ such that $\Theta_1 \leq \Theta_2$. By Theorem 4.6, there is a surjective homomorphism $f : H/\Theta_1 \to H/\Theta_2$ given by $f([x]_{\Theta_1}) = [x]_{\Theta_2}$. Since $V(\Gamma(H/\Theta_1)) = H/\Theta_1$ and $V(\Gamma(H/\Theta_2)) = H/\Theta_2$, there exists $\overline{f} : V(\Gamma(H/\Theta_1)) \to V(\Gamma(H/\Theta_1))$ given by $\overline{f}([x]_{\Theta_1}) = f([x]_{\Theta_1})$. Also, by Corollary 4.7, there exists a surjective homomorphism

$f: H/\Theta_H \to H/\Theta$ for all $\Theta \in Con_R(H)$ given by $f([x]_{\Theta_H}) = [x]_\Theta$. Since $V(\Gamma(H/\Theta_H)) = H/\Theta_H$ and $V(\Gamma(H/\Theta)) = H/\Theta$, there exists $f': V(\Gamma(H/\Theta_H)) \to V(\Gamma(H/\Theta))$ given by $f'([x]_{\Theta_H}) = f([x]_{\Theta_H})$.

*Theorem 4.8 Let $H$ be a hyper BCK-algebra and let $\Theta_1, \Theta_2 \in Con_R(H)$ such that $\Theta_1 \leq \Theta_2$. If $[x]_{\Theta_1}$ and $[y]_{\Theta_1}$ are adjacent vertices on $\Gamma(H/\Theta_1)$, then either $\overline{f}([x]_{\Theta_1}) = \overline{f}([y]_{\Theta_1})$ or $\overline{f}([x]_{\Theta_1})$ and $\overline{f}([y]_{\Theta_1})$ are adjacent vertices on $\Gamma(H/\Theta_2)$.*

*Proof*: Let $[x]_{\Theta_1}[y]_{\Theta_1} \in E(\Gamma(H/\Theta_1))$ and suppose $\overline{f}([x]_{\Theta_1}) \neq \overline{f}([y]_{\Theta_1})$. Then $[x]_{\Theta_2} \neq [y]_{\Theta_2}$. By Lemma 4.2, $[x]_{\Theta_2}[y]_{\Theta_2} \in E(\Gamma(H/\Theta_2))$. Therefore, $\overline{f}([x]_{\Theta_1})\overline{f}([y]_{\Theta_1}) \in E(\Gamma(H/\Theta_2))$. Now, suppose $\overline{f}([x]_{\Theta_1})\overline{f}([y]_{\Theta_1}) \notin E(\Gamma(H/\Theta_2))$. Then either $\overline{f}([x]_{\Theta_1}) = \overline{f}([y]_{\Theta_1})$ or $[z]_{\Theta_2} \in L_{V(H/\Theta_2)}(\{\overline{f}([x]_{\Theta_1}), \overline{f}([y]_{\Theta_1})\})$ where $[z]_{\Theta_2} \neq [0]_{\Theta_2}$. Suppose $[z]_{\Theta_2} \in L_{V(H/\Theta_2)}(\{\overline{f}([x]_{\Theta_1}), \overline{f}([y]_{\Theta_1})\})$ where $[z]_{\Theta_2} \neq [0]_{\Theta_2}$. Since $[x]_{\Theta_1}[y]_{\Theta_1} \in E(\Gamma(H/\Theta_1))$, by Lemma 4.2, $[x]_{\Theta_2}[y]_{\Theta_2} \in E(\Gamma(H/\Theta_2))$. This is a contradiction. Therefore, $\overline{f}([x]_{\Theta_1}) = \overline{f}([y]_{\Theta_1})$. ∎

*Corollary 4.9 Let $H$ be a hyper BCK-algebra, $\Theta_H = \{(x,x) \mid x \in H\}$ and $\Theta \in Con_R(H)$. If $[x]_{\Theta_H}$ and $[y]_{\Theta_H}$ are adjacent vertices on $\Gamma(H/\Theta_H)$, then either $\overline{f}([x]_{\Theta_H}) = \overline{f}([y]_{\Theta_H})$ or $\overline{f}([x]_{\Theta_H})$ and $\overline{f}([y]_{\Theta_H})$ are adjacent vertices on $\Gamma(H/\Theta)$.*

*Theorem 4.10 Let $H$ be a hyper BCK-algebra and let $\Theta_1, \Theta_2 \in Con_R(H)$ such that $\Theta_1 \leq \Theta_2$. Then there exists an injective graph homomorphism $g: V(\Gamma(H/\Theta_2)) \to V(\Gamma(H/\Theta_1))$ such that $\overline{f}g = 1_{V(\Gamma(H/\Theta_2))}$. Furthermore, $\Gamma(H/\Theta_2)$ is isomorphic to a subgraph of $\Gamma(H/\Theta_1)$.*

*Proof*: Let $[x]_{\Theta_2} \in H/\Theta_2$ with $x_0 \in [x]_{\Theta_2}$. Define $g: V(\Gamma(H/\Theta_2)) \to V(\Gamma(H/\Theta_1))$ by $g([x]_{\Theta_2}) = [x_0]_{\Theta_1}$. We will show that $g$ is an injective function. Let $[x]_{\Theta_2}, [y]_{\Theta_2} \in V(\Gamma(H/\Theta_2))$ with $x_0 \in [x]_{\Theta_2}$ and $y_0 \in [y]_{\Theta_2}$. Then $g([x]_{\Theta_2}) = [x_0]_{\Theta_1}$ and $g([y]_{\Theta_2}) = [y_0]_{\Theta_1}$. Suppose $[x]_{\Theta_2} = [y]_{\Theta_2}$. If $x_0 \in [x]_{\Theta_2}$, then $x_0 \in [y]_{\Theta_2}$. Thus, $g([x]_{\Theta_2}) = [x_0]_{\Theta_1} = g([y]_{\Theta_2})$. Hence, $g$ is well-defined. Let

$[x]_{\Theta_2}, [y]_{\Theta_2} \in V(\Gamma(H/\Theta_2))$ such that $g([x]_{\Theta_2}) = g([y]_{\Theta_2})$. If $x_0 \in [x]_{\Theta_2}$ and $y_0 \in [y]_{\Theta_2}$, then $[x]_{\Theta_2} = [x_0]_{\Theta_2} = [y_0]_{\Theta_2} = [y]_{\Theta_2}$. Hence, $g$ is injective. We will show that $g$ is a graph homomorphism. Let $[x]_{\Theta_2}, [y]_{\Theta_2}, [0]_{\Theta_2} \in V(\Gamma(H/\Theta_2))$ with $x_0 \in [x]_{\Theta_2}$, $y_0 \in [y]_{\Theta_2}$ and $0 \in [0]_{\Theta_2}$ such that $[x]_{\Theta_2}[y]_{\Theta_2} \in E(\Gamma(H/\Theta_2))$. Then $L_{H/\Theta_2}(\{[x]_{\Theta_2}, [y]_{\Theta_2}\}) = \{[0]_{\Theta_2}\}$. Thus, $[w]_{\Theta_2} \not\ll [x]_{\Theta_2}$ and $[w]_{\Theta_2} \not\ll [y]_{\Theta_2}$ for all $w \in H \setminus \{[0]_{\Theta_2}\}$. Hence, $[0]_{\Theta_2} \notin [w]_{\Theta_2} * [x]_{\Theta_2}$ and $[0]_{\Theta_2} \notin [w]_{\Theta_2} * [y]_{\Theta_2}$. By Theorem 2.4, $0 \notin w * x$ and $0 \notin w * y$. This implies that $[0]_{\Theta_1} \notin [w]_{\Theta_1} * [x]_{\Theta_1}$ and $[0]_{\Theta_1} \notin [w]_{\Theta_1} * [y]_{\Theta_1}$. So, $[w]_{\Theta_1} \not\ll [x]_{\Theta_1}$ and $[w]_{\Theta_1} \not\ll [y]_{\Theta_1}$ for all $w \in H \setminus \{[0]_{\Theta_1}\}$. Since $H/\Theta_1$ is a hyper BCK-algebra, by Theorem 2.2 (a1), $[0]_{\Theta_1} \ll [x]_{\Theta_1}$ and $[0]_{\Theta_1} \ll [y]_{\Theta_1}$. Hence, $L_{H/\Theta_1}(\{[x_0]_{\Theta_1}, [y_0]_{\Theta_1}\}) = \{[0]_{\Theta_1}\}$. Thus, $[x_0]_{\Theta_1}[y_0]_{\Theta_1} \in E(\Gamma(H/\Theta_1))$. Therefore, $g$ is a graph homomorphism. Hence, $\Gamma(H/\Theta_2)$ is isomorphic to a subgraph of $\Gamma(H/\Theta_1)$. Furthermore, $f(g([x]_{\Theta_2})) = f([x_0]_{\Theta_1}) = [x_0]_{\Theta_2} = [x]_{\Theta_2}$. Consequently, $fg = 1_{\Gamma(H/\Theta_2)}$. ■

*Corollary 4.11 Let* $H$ *be a hyper BCK-algebra and let* $\Theta_H = \{(x,x) \mid x \in H\}$. *Then there exists an injective graph homomorphism* $g : V(\Gamma(H/\Theta)) \to V(\Gamma(H/\Theta_H))$ *such that* $f'g = 1_{V(\Gamma(H/\Theta))}$. *Furthermore,* $\Gamma(H/\Theta)$ *is isomorphic to a subgraph of* $\Gamma(H/\Theta_H)$.

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