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Multiple Congruences of Lattices By Gluing

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Abstract

In this paper we use the Hall – Dilworth gluing construction to obtain multiple congruences of a lattice L. For any finite lattice L, $G_m(L, B_n)$, the gluing of L and B_n over F and I, both F and I are isomorphic to C_2. For any lattice L, the congruences of $G_m(L, B_n)$ is $2^{(n-1)}$ times the congruences of L where F be the filter of L and I be an ideal of B_n and are isomorphic to C_2. We call $G_m(L, B_n)$ the congruence multiple operator.

Key words : Gluing, filter, ideal, congruence and multiple congruence

Mathematics Subject classification: Primary: 06B10, Secondary: 06D05

1 Introduction

In[1,2,3] Gratzer and etal studied congruence lattices of lattices. The gluing construction in the lattice theory started with a paper of M. Hall and R.P. Dilworth[4] to prove that there exists a modular lattices that cannot be embedded in any complemented modular lattice. This construction is as follows.

Let K and L be lattices. Let F be filter of K and let I be an ideal of L. If F is isomorphic to I with ψ as the isomorphism, then we can form the gluing of K and L over F and I with respect to ψ defined as follows.

We form the disjoint union K∪L and identify a∈F, to obtain the set G. we order G as follows:

$$a \le b \text{ if and only if, } \begin{cases} a \le_K b \text{ if } a, b \in K \\ a \le_L b \text{ if } a, b \in L \\ a \le_K x \text{ and } \psi(x) \le_L b \text{ if } a \in K \text{ and } b \in L \text{ for some } x \in F \end{cases}.$$

Lemma 1 [5]:

G is a lattice. The join in G is described by

$$a \vee_G b = \begin{cases} a \vee_K b \text{ if } a, b \in K \\ a \vee_L b \text{ if } a, b \in L \\ \psi(a \vee_K x) \vee_L b \text{ if } a \in K, b \in L \text{ for any } x \in F \text{ and } x \leq b \end{cases}$$

and dually for the meet. If L has a zero, 0_L, then the last clause for the join may be rephrased: $a \vee_G b = \psi(a \vee_K 0_L) \vee_L b$ if $a \in K$ and $b \in L$. G contains K and L as sublattices. Infact, K is an ideal and L is a filter of G.

Lemma 2[5]*:*

Let K, L, F, I and G be given as above. Let A be a lattice containing K and L as sublattices so that $K \cap L = I = F$. Then $K \cup L$ is a sublattice of A and it is isomorphic to G.

Definition 3[5] *:*

If θ_K is a binary relation on K and θ_L is a binary relation on L, the reflexive product $\theta_K \overset{r}{\circ} \theta_L$ is defined as $\theta_K \cup \theta_L \cup (\theta_K \circ \theta_L)$.

Lemma 4[5]*:* A congruence θ of G can be uniquely written in the form $\theta = \theta_K \overset{r}{\circ} \theta_L$, θ_K where θ_K is a congruence of K and θ_L is a congruence of L satisfying the condition that θ_K restricted to F equals θ_L restricted to I (under the identification of elements by ψ). Conversely, if θ_K is a congruence of K and θ_L is a congruence of L satisfying the condition that θ_K restricted to F equals θ_L restricted to I, then $\theta = \theta_K \overset{r}{\circ} \theta_L$ is a congruence of G.

Lemma 5[5]*:*

Let G be the gluing of lattices K and L over F and I as above. If K and L are modular so is the gluing G of K and L. If K and L are distributive, so is the gluing G.

In this paper, we study about the multiple congruences of a lattice L. For any finite lattice L, $G_m(L, B_n)$, the gluing of L and B_n over F and I, both F and I are isomorphic to C_2. For any lattice L, the congruences of $G_m(L, B_n)$ is $2^{(n-1)}$ times the congruences of L where F be the filter of L and I be an ideal of B_n and are isomorphic to C_2. We call $G_m(L, B_n)$ the congruence multiple operator. In[6], we studied about proper modular congruence preserving extensions lattices.

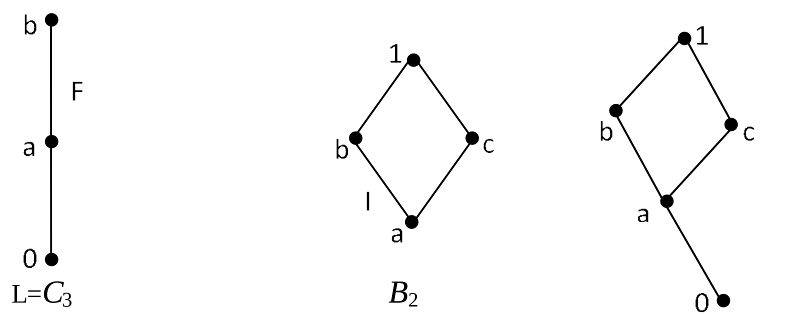
3 Multiple Congruence of a lattice :

Definition 6 : The lattice G is called the congruence multiple operator if $Con G \cong C_2 \times Con L$ for all

L. Equivalently for any lattice L, the congruence of an extension of L is multiple.

Construction 7 : Let L be a finite bounded lattice with filter F and let B_2 be a nontrivial finite non simple lattice with an ideal I of B_2 isomorphic to C_2. B_2 is not a congruence preserving extension of I. Now glue the lattices L and B_2 over F and I under the isomorphism ψ. Let us denote it by $G_m(L,B_2)$.

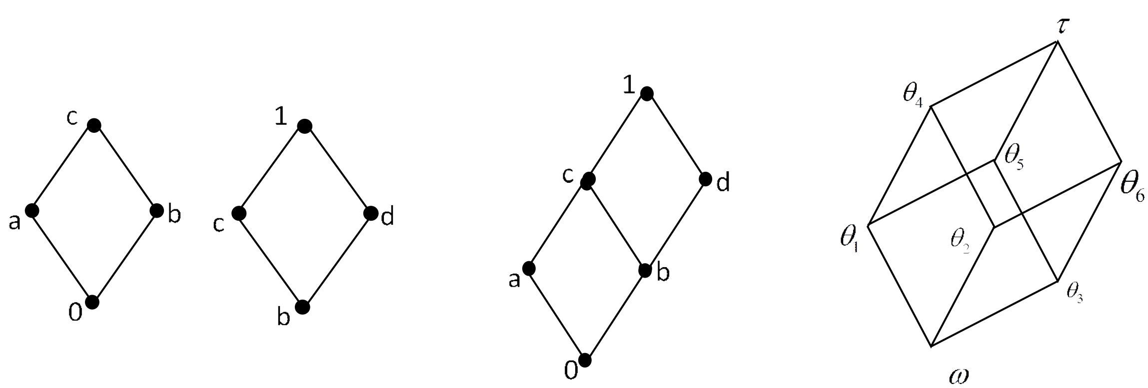
Example 8: Let L be a chain with three elements with a filter F isomorphic to C_2 and let B_2 be a Boolean algebra with two atoms having an ideal I isomorphic to F. Glue the lattices L and B_2 over the filter F of L and an ideal I of B_2. The lattices L, B_2 and $G_m(L,B_2)$ are shown in figure 1.



L=C_3 B_2 $G_m(L=C_3,B_2)$

Figure 1

Also their congruence with the congruence classes are given below. The congruence lattices of C_3. B_2, and G(Con(C_3), Con(B_2)) are given in figure 2.



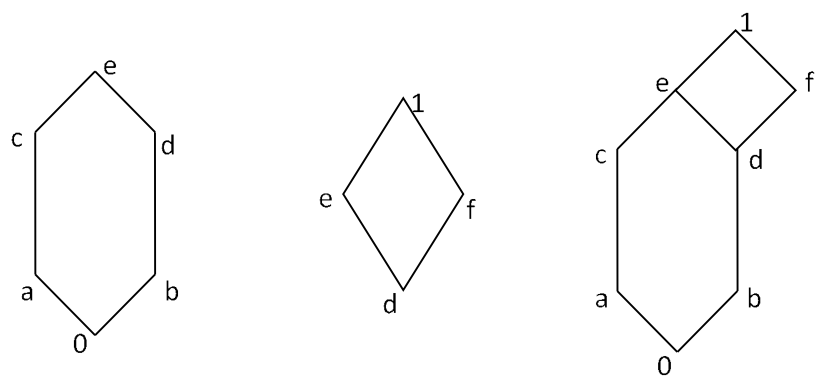
Con(C_3) Con(B_2) G(Con(C_3),Con(B_2)) Con(G(Con(C_3),Con(B_2)))

Figure 2

The congruence of C_3 is Con(C_3)= { $\omega,\theta_1,\theta_2,\tau$ } where ω- the null congruence, θ_1={(0),(a,1)}, θ_2={(0,a),(1)}and τ- all congruence. The congruence lattice of C_3 is isomorphic to B_2. Glue the lattices Con(C_3) and Con(B_2) over the filter F and an ideal I both isomorphic to C_2.

The congruence of B_2 is Con(B_2) = { $\omega, \theta_1, \theta_2, \tau$ } where ω- the null congruence, θ_1 ={(a,b),(c,1)}, θ_2={(a,c), (b,1)}and τ- all congruence. The congruence lattice of B_2 is isomorphic to B_2. The lattice G_m (C_3, B_2) is obtained by gluing C_3 and B_2 over the filter F of C_3 and an ideal I of B_2. The congruence of G_m (C_3, B_2) is obtained by the reflexive binary product of congruences of C_3 and congruences of B_2. ConG_m (C_3, B_2))={ $\omega, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \tau$ }, where ω - the null congruence, θ_1 ={(0,a),(b),(c,1)}; θ_2={(0),(a,c),(b,1)}; θ_3={(0)(a,b)(c,1)}; θ_4={(0ab)(c1)}; θ_5={(0ac)(b1)}; $\theta_6 = \{(0)(abc1)\}$ and τ- all congruence. The congruence lattice of G(C_3, B_2) is isomorphic to B_2. That is, every congruence of C_3 is restricted to two times of the congruences of B_2.

Example 9: Let L be an ortholattice. Let G_m (C_3, B_2) be the gluing of L and B_2 over F and I under the isomorphism ψ. The lattice L, G_m (L, B_2) are given in figure 3.



a) L (b) B_2 (c) G_m (L, B_2)

Figure 3

The congruence of G_m (L, B_2) is given by Con (G_m (L, B_2)) isomorphic to $ConL \times C_2$. Con(G_m (L, B_2))= {ω, τ, θ's}. Where ω is the null congruence, τ is the all congruence and θ's the non trivial congruence of G_m (L, B_2). They are {{(0),(a),(b),(c),(df),(e1)}, {(0),(ac),(bd),(e),(f),(1)}, {(0),(ac),(bdf),(e1)}, {(0a),(b),(c),(de),(f1)}, {(0a),(b),(c),(def1)}, {(0b),(ce),(a),(d),(f),(1)}, {(0b),(ce1),(a),(df)}, {(0ac),(bde),(f1)}, {(0ac),(bdef1)}, {(0bd),(ace),(f),(1)}, {(0bdf),(ace1)}, {0abcde),(f1)} }. As there are seven congruence of L, every nontrivial congruence of B_2 restricted to an ideal I equal twice the congruence of L restricted to a filter F. The congruence lattice of L and G_m (L, B_2) is given in figure 4.

Hence by gluing of a Boolean algebra with an ortholattice L, whose ideal and filter are simple lattice (C_2) has 14 congruences (2 X 7) isomorphic to $C_2 \times$ Con(L).

Hence it is isomorphic to two times the congruence of L.

Remark 10: From the construction given in I, the lattice G_m (L, B_2) contains a sub lattice L, which is an ideal of G_m (L, B_2) .

Remark 11: The number of congruences of B_n , Boolean algebra with n atoms is 2^n. If all the congruences

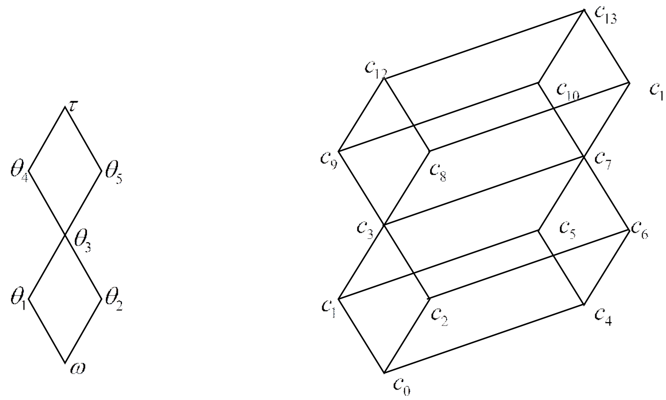


Figure 4

θ_B of B_n is restricted to a prime ideal I=$(a]$ of B_n, then $\theta_B \cap I$ = $(0)(a)$ or $(0a)$. Therefore the number of congruences of B_n containing $(0)(a)$ (or) $(0a)$ is 2^{n-1}.

Theorem 12: Let L be any lattice with a prime filter F. Then the number of congruences of $G_n(L)$ is 2^{n-1} times the number of congruences of L.

Proof. Let L be any finite lattice with a prime filter F and B_n be a Boolean algebra with n atoms. Let I=$(a]$ be a prime ideal of B_n and F be a filter of L which is isomorphic to I under an isomorphism ψ. Let θ_L be a congruence of L. Then $\theta_L \cap F$ is either equal to $\{(0)(a)\}$ or $\{(0a)\}$. If $\theta_L \cap F$ is equal to 2^{n-1} congruences of the gluing of B_n restricted to I. By definition 3, $\theta = \theta_L \circ^r \theta_{B_n}$

is a congruence of $G_n(L)$. Therefore corresponding to each θ_L, there are 2^{n-1} θ_{B_n} such that $\theta_L \cap F = \theta_{B_n} \cap I$ and $\theta = \theta_L \circ^r \theta_{B_n}$ is a congruence of $G_n(L)$. Hence $|Con(G_n(L))| = 2^{n-1} \times |Con(L)|$.

Conclusion

From the gluing of lattices we obtain that for every finite lattice A and a Boolean algebra B, the congruence of gluing of A and B over F and I is isomorphic to 2^{n-1} times the congruence of A. Congruence of gluing of A and B over F and I is isomorphic to gluing of con(A) and con(B) over F and I under ψ, where F and I of A, B are same as F and I of Con(A) and Con(B). In[7] I have studied about Distributive congruence preserving extension of chains using relative separator. In[6] we studied about proper modular congruence preserving extension of lattices by gluing of simple modular lattice. We further extend the gluing concept to get a proper congruence preserving extension of A if B is not a congruence preserving extension of I.

References

1. G. Gratzer and H. Lakser, Congruence lattices of planar lattices, *Acta Math. Hungar. 60,* 251-268 (1992).

2. G. Gratzer, H. Lakser, and E.T. Schmidt, Congruence lattices of small planar lattices, *Proc. Amer. Math. Soc. 123*, 2619-2623 (1995).
3. G. Gratzer and E.T. Schmidt, Congruence Lattices of lattices, *Acta Math. Acad Sci*. Hungar *13*, 179-185 (1962).
4. M. Hall and R. P. Dilworth, The embedding problem for modular lattices, Annals Math., *45*, 450-456 (1944).
5. G. Gratzer, "The Congruences of a finite lattice", A proof-by-picture approach, Birkhauser Boston (2006).
6. E.K.R. Nagarajan and R. Rathajeyalakshmi, "Modular Congruence preserving extension of lattices", *Ultra Scientist of Physical Sciences*, vol *21*, No.1(M), 173-176 (2009).
7. R. Ratha Jeyalakshmi, "Distributive Congruence preserving extensions of Chains", International Journal of Science Technology and Engineering, vol 3 (issue 7), 174-180 (2017).