ISSN 2231-346X (Print)
JUSPS-A Vol. 29(9), 401-409 (2017). Periodicity-Monthly
Section A
(Online) ISSN 2319-8044

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

Indexed Absolute Riesz Summability Using General Class of Power Increasing Sequence

D. ACHARYA [1], S.SAHU [2*], P.C.NAYAK [3] and U.K.MISRA [4]

[1]National Institute of Science and Technology, Ganjam, Odisha, (India)
[2]National Institute of Science and Technology, Ganjam, Odisha, India
[3]Bhadrak(Autonomous) College, Bhadrak, Odisha, India
[4]National Institute of Science and Technology, Ganjam, Odisha, India
Email Address of Corresponding Author: E-mail: dpak.acharya888@gmail.com,
* Correspondence author E-mail: subrataviit@gmail.com
http://dx.doi.org/10.22147/jusps-A/290906

Acceptance Date 13th August, 2017, Online Publication Date 2nd September, 2017

Abstract

Extending the result of Bor(2016) and subsequently Majhi et al, a new result concerning absolute indexed Riesz Summability factors, using quasi power increasing sequence, has been established.

Key words : Riesz mean, Summability factors, Quasi-increasing; Quasi - f - power increasing; indexed absolute summability.
Subject Classification 26D45, 40A05, 40D15, 40F05

1. Introduction

A sequence (a_n) of positive numbers is said to be almost increasing if there exists a positive sequence (b_n) and two positive constants A and B such that

(1.1) $Ab_n \le a_n \le Bb_n$, for all $n \in N.$

For $0 < \beta < 1$, it is said to be quasi-β-power increasing, if there exists a constant K depending upon β with $K \ge 1$ such that

(1.2) $K n^{\beta} a_n \geq m^{\beta} a_m$, for all $n \geq m$.

In particular if $\beta = 0$, then (a_n) is a quasi-increasing sequence. It is clear that for any non-negative β, every almost increasing sequence is a quasi-β-power increasing sequence. But the converse is not true in general, as $(n^{-\beta})$ is quasi-β-power increasing but not almost increasing.

Let $f = (f_n)$ be a positive sequence of numbers. Then the positive sequence (a_n) is said to be quasi-f-power increasing, if there exists a constant K depending upon f with $K \geq 1$ such that

(1.3) $K\, f_n a_n \geq f_m a_m$

for $n \geq m \geq 1$. Clearly, if (a_n) is a quasi-f-power increasing sequence, then $(a_n\, f_n)$ is also a quasi- increasing sequence.

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n) and let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \text{ as } n \to \infty .$$

The sequence to sequence transform

(1.4) $$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu\, s_\nu \;,\; P_n \neq 0,$$

defines the sequence $\{t_n\}$ of the $(\overline{N}, p_n)$- mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see[3]).

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k, k \geq 1$ [1], if

(1.5) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty .$$

Let $\{\theta_n\}$ be any sequence of positive constants. The seriesis $\sum a_n$ said to be summable $\left|\overline{N}, p_n, \theta_n\right|_k, k \geq 1$[8], if

(1.6) $$\sum_{n=1}^{\infty} (\theta_n)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty .$$

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n, \theta_n, \mu\right|_k, k \geq 1, \mu \geq 0$, if

(1.7) $$\sum_{n=1}^{\infty} (\theta_n)^{\mu k + k - 1} \left|t_n - t_{n-1}\right|^k < \infty .$$

If we take $\mu = 0,$ then $\left|\overline{N}, p_n, \theta_n, \mu\right|_k -$ summability reduces to $\left|\overline{N}, p_n, \theta_n\right|_k -$ summability.

If we take $\theta_n = \dfrac{P_n}{p_n}$ and $\mu = 0$ then $\left|\overline{N}, p_n, \theta_n, \mu\right|_k -$ summability reduces to $\left|\overline{N}, p_n\right|_k -$ summability.

For any real γ, the series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n, \theta_n, \mu, \gamma\right|_k, k \geq 1, \mu \geq 0,$ if

(1.8) $$\sum_{n=1}^{\infty} (\theta_n)^{\gamma(\mu k + k - 1)} \left|t_n - t_{n-1}\right|^k < \infty .$$

For $\gamma = 1$, $\left|\overline{N}, p_n, \theta_n, \mu, \gamma\right|_k -$ summability reduces to $\left|\overline{N}, p_n, \theta_n, \mu\right|_k -$ summability .

2. Known Theorems :

Dealing with $\left|\overline{N}, p_n, \theta_n\right|_k -$ summability factors using a new general class of power increasing sequences Bor[2] proved the following result.

2.1. Theorem - A

Let $f = (f_n)$ be a sequence where $f_n = n^{\sigma}(\log n)^{\eta}$, $\eta \geq 0,\ 0 < \sigma < 1$. Let $(\lambda_n) \in BV$ and (X_n) be a quasi-f-power increasing sequence. Suppose also that there exists sequences (β_n) and (λ_n) such that

(2.1.1) $\left|\Delta\lambda_n\right| \leq \beta_n$,

(2.1.2) $\beta_n \to 0 \ as\ n \to \infty$,

(2.1.3) $\sum_{n=1}^{m} n\left|\Delta\beta_n\right| X_n < \infty$,

(2.1.4) $\left|\lambda_n\right| X_n = O(1)$

and $\{p_n\}$ is a sequence such that

(2.1.5) $P_n = O(np_n)$

(2.1.6) $P_n \Delta p_n = O(p_n p_{n+1})$

(2.1.7) $\sum_{\nu=1}^{n} \theta_{\nu}^{k-1} \nu^{-k} \left|s_{\nu}\right|^k = O(X_n),\ as\ n \to \infty$

are satisfied and $\dfrac{\theta_n p_n}{P_n}$ is a non-increasing sequence, then the series $\sum a_n \dfrac{P_n \lambda_n}{np_n}$ is summable $\left|\overline{N}, p_n, \theta_n\right|_k, k \geq 1$.

Subsequently, extending theorem –A, Majhi *et al.*,[5] has established the following theorem:

2.2. Theorem – B

Let $f=(f_n)$ be a sequence where $f_n=n^{\sigma}(\log n)^{\eta}$, $\eta\geq 0,\ 0<\sigma<1$. Let $(\lambda_n)\in BV$ and (X_n) be a quasi- f – -power increasing sequence. Suppose that there exists sequences (β_n) and (λ_n) such that

(2.2.1) $|\Delta\lambda_n|\leq\beta_n$,

(2.2.2) $\beta_n\to 0\ as\ n\to\infty$,

(2.2.3) $\sum_{n=1}^{m} n|\Delta\beta_n|X_n<\infty$,

(2.2.4) $|\lambda_n|X_n=O(1)$

Further let (p_n) be a sequence such that

(2.2.5) $P_n=O(np_n)$

(2.2.6) $P_n\Delta p_n=O(p_n p_{n+1})$

(2.2.7) $\sum_{v=1}^{n}\theta_v^{\mu k+k-1}v^{-k}|s_v|^k=O(X_n)\ as\ n\to\infty$

are satisfied and $\frac{\theta_n p_n}{P_n}$ be a non-increasing sequence. hen the series $\sum a_n\frac{P_n\lambda_n}{np_n}$ is summable $\left|\overline{N},p_n,\theta_n,\mu\right|_k, k\geq 1, \mu>0.$

3. Main Result

In what follows in this paper, we proved a result concerning absolute indexed Riesz Summability $\left|\overline{N},p_n,\theta_n,\mu,\rho\right|_k$, $k\geq 1,\mu>0,\rho\geq 0$ of a factored series using f – power increasing sequence. We prove:

Theorem 3.1

Let $f=(f_n)$ be a sequence, where $f_n=n^{\sigma}(\log n)^{\eta}$, $\eta\geq 0,\ 0<\sigma<1$. Let $(\lambda_n)\in BV$ and (X_n) be a quasi- f – -power increasing sequence. Suppose that there exists sequences (β_n) and (λ_n) such that

(3.1.1) $|\Delta\lambda_n|\leq\beta_n$,

(3.1.2) $\beta_n\to 0\ as\ n\to\infty$,

(3.1.3) $\sum_{n=1}^{m} n|\Delta\beta_n|X_n < \infty$,

(3.1.4) $|\lambda_n|X_n = O(1)$

Further let (p_n) be a sequence such that

(3.1.5) $P_n = O(np_n)$

(3.1.6) $P_n \Delta p_n = O(p_n p_{n+1})$

(3.1.7) $\sum_{\nu=1}^{n} \theta_\nu^{\rho(\mu k+k-1)} \nu^{-k} |s_\nu|^k = O(X_n)$ *as* $n \to \infty$

are satisfied and $\frac{\theta_n p_n}{P_n}$ be a non-increasing sequence. hen the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $\left|\overline{N}, p_n, \theta_n, \mu, \rho\right|_k, k \geq 1, \mu > 0, \rho > 0$

In order to prove the theorem we require the following lemmas.

4.1. Lemma[7]

Under the conditions on (X_n), (β_n) and (λ_n) as prescribed in the statement of the theorem

(4.1.1) $nX_n\beta_n = O(1)$

and

(4.1.2) $\sum_{n=1}^{\infty} \beta_n X_n < \infty$.

4.2. Lemma[6]

If the conditions (3.1.5) and (3.1.6) are satisfied then

(4.2.1) $\Delta\left(\frac{P_n}{p_n}\right) = O\left(\frac{1}{n}\right)$

5. Proof of the Theorem

Let (T_n) be the sequence of $(\overline{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then by the definition, we have

$$T_n = \frac{1}{P_n}\sum_{\upsilon=1}^{n} p_\upsilon \sum_{r=1}^{\upsilon} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n}\sum_{\upsilon=1}^{n} (P_n - P_{\upsilon-1}) \frac{a_\nu P_\nu \lambda_\nu}{\nu p_\nu}$$

Hence for $n \geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\upsilon=1}^{n} \frac{P_{\upsilon-1} P_\nu a_\nu \lambda_\nu}{\nu p_\nu}$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\upsilon=1}^{n-1} s_\nu \, \Delta\left(\frac{P_{\nu-1} P_\nu \lambda_\nu}{\nu p_\nu}\right) + \frac{\lambda_n s_n}{n}$$

$$= \frac{\lambda_n s_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} s_\nu \frac{P_{\nu+1} P_\nu \Delta\lambda_\nu}{(\nu+1) p_{\nu+1}} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu s_\nu \lambda_\nu \Delta\left(\frac{P_\nu}{\nu p_\nu}\right) - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu s_\nu \lambda_\nu \left(\frac{1}{\nu}\right)$$

$= T_{n1} + T_{n2} + T_{n3} + T_{n4}$ (say).

In order to prove the theorem, using Minkowski's inequality it is enough to show that

$$\sum_{n=1}^{\infty} (\theta_n)^{\rho(\mu k+k-1)} \left|T_{n,r}\right|^k < \infty, r = 1, 2, 3, 4.$$

Applying Hölder's inequality, we have

$$\sum_{n=1}^{m} (\theta_n)^{\rho(\mu k+k-1)} \left|T_{n,1}\right|^k = \sum_{n=1}^{m} (\theta_n)^{\rho(\mu k+k-1)} n^{-k} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k$$

$$= O(1) \sum_{n=1}^{m} (\theta_n)^{\rho(\mu k+k-1)} |\lambda_n| n^{-k} |s_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{\nu=1}^{n} (\theta_\nu)^{\rho(\mu k+k-1)} \nu^{-k} |s_\nu|^k + O(1) |\lambda_m| \sum_{n=1}^{m} (\theta_n)^{\rho(\mu k+k-1)} n^{-k} |s_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m, \; by \; (3.1.7)$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m, \; by \; (3.1.1)$$

$$= O(1) \text{ as } m \to \infty., \text{by lemma 4.1 and } (3.1.4)$$

Next,

$$\sum_{n=2}^{m+1} (\theta_n)^{\rho(\mu k+k-1)} \left|T_{n,2}\right|^k = O(1) \sum_{n=2}^{m+1} (\theta_n)^{\rho(\mu k+k-1)} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left|\sum_{\nu=1}^{n-1} P_\nu s_\nu \Delta\lambda_\nu\right|^k$$

$$= O(1) \sum_{n=2}^{m+1} (\theta_n)^{\rho(\mu k+k-1)} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{\sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} |s_\nu| |p_\nu| |\Delta\lambda_\nu|\right\}^k$$

$$\leq O(1)\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}\left(\frac{P_\nu}{p_\nu}\right)^k |s_\nu|^k\, p_\nu\,(\beta_\nu)^k\left\{\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_\nu\right\}^{k-1}\text{, by } (3.1.1)$$

$$= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^k |s_\nu|^k\, p_\nu\,(\beta_\nu)^k(\theta_\nu)^{\rho(\mu k+k-1)}\left(\frac{p_\nu}{P_\nu}\right)^{k-1}\sum_{n=\nu+1}^{m+1}\left\{\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}\frac{p_n}{P_nP_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^k |s_\nu|^k(\beta_\nu)^k\left(\frac{p_\nu}{P_\nu}\right)(\theta_\nu)^{\rho(\mu k+k-1)}\left(\frac{p_\nu}{P_\nu}\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}(\nu\beta_\nu)^{k-1}(\nu\beta_\nu)\frac{1}{\nu^k}(\theta_\nu)^{\rho(\mu k+k-1)}|s_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m}(\nu\beta_\nu)\nu^{-k}(\theta_\nu)^{\rho(\mu k+k-1)}|s_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m}\Delta(\nu\beta_\nu)\sum_{r=1}^{\nu}(\theta_r)^{\rho(\mu k+k-1)}r^{-k}|s_r|^k + O(1)(m\beta_m)\sum_{\nu=1}^{m}(\theta_\nu)^{\rho(\mu k+k-1)}\nu^{-k}|s_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m}|\Delta(\nu\beta_\nu)|X_\nu + O(1)(m\beta_m)X_m\text{ , by } (3.1.7)$$

$$= O(1)\sum_{\nu=1}^{m-1}\nu|\Delta\beta_\nu|X_\nu + O(1)\sum_{\nu=1}^{m-1}\beta_\nu X_\nu + O(1)(m\beta_m)X_m$$

$= O(1)$ as $m\to\infty$.

Again,

$$\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}|T_{n,3}|^k = O(1)\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left\{\sum_{\nu=1}^{n-1}P_\nu|s_\nu|\frac{|\lambda_\nu|}{\nu}\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}\left(\frac{P_\nu}{p_\nu}\right)^k\nu^{-k}p_\nu|s_\nu|^k|\lambda_\nu|^k\left\{\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_\nu\right\}^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^k|s_\nu|^k\,p_\nu\,(\nu)^{-k}|\lambda_\nu|^k\sum_{n=\nu+1}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)$$

$$= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^{k-1}(\nu)^{-k}(\theta_\nu)^{\rho(\mu k+k-1)}\left(\frac{p_\nu}{P_\nu}\right)^{k-1}|\lambda_\nu|^{k-1}|\lambda_\nu||s_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|(\theta_\nu)^{\rho(\mu k+k-1)}(\nu)^{-k}|s_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m-1}\beta_\nu X_\nu + O(1)|\lambda_m|X_m$$

$$= O(1) \text{ as } m \to \infty.$$

Finally,

$$\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}|T_{n,4}|^k = \sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k}\left|\sum_{\nu=1}^{n-1}P_\nu s_\nu \frac{\lambda_\nu}{\nu}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k}\left|\sum_{\nu=1}^{n-1}P_\nu s_\nu\, p_\nu \frac{\lambda_\nu}{\nu\, p_\nu}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}(\theta_n)^{\rho(\mu k+k-1)}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}\left(\frac{P_\nu}{p_\nu}\right)^k |s_\nu|^k \nu^{-k}\, p_\nu |\lambda_\nu|^k \left\{\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} p_\nu\right\}^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^k \nu^{-k}|s_\nu|^k\, p_\nu |\lambda_\nu|^k \frac{1}{P_\nu}(\theta_\nu)^{\rho(\mu k+k-1)}\left(\frac{p_\nu}{P_\nu}\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}(\theta_\nu)^{\rho(\mu k+k-1)}\nu^{-k}|s_\nu|^k|\lambda_\nu|$$

$$= O(1)\sum_{\nu=1}^{m-1}\beta_\nu X_\nu + O(1)|\lambda_m|X_m$$

as $= O(1)$ as $m \to \infty.$

This completes the proof of the theorem.

6. Conclusion

Our Theorem generalizes Theorem-A and Theorem-B. Putting $\rho = 1$, Theorem-B becomes a particular case of our Theorem. Putting $\mu = 0$ *and* $\rho = 1$, Theorem-A becomes a particular result of our Theorem. One can extend our result for Indexed *Nörlund Summability* with different parameters.

References

1. Bor, H., two summability methods, *Math. Proc. Camb. Philos. Soc., 97,* 147-149 (1985).

2. Bor, H., On quasi-power increasing sequences and their some applications, *International Journal of Analysis and Applications*, Vol-*10(2)*, 95-100 (2016).
3. Hardy, G. H., Divergent Series, First edition, Clarendon press, Oxford University. (1949).
4. Leindler, L., A new application of quasi power increasing sequences, Publ. Math. Debrecen, *58*, 791-796 (2001).
5. Majhi, B., Padhy, B., Samanta, P., Misra, M. and Misra,U.K., A new theorem on indexed absolute Riesez summability using power increasing sequence, communicated to Bulletin of Pure and Applied sciences.
6. Mishra, K. N. and Srivastava, R. S. L., On summability factors of infinite series, *Indian J. Pure Appl. Math.*, *15*, 651-656 (1984).
7. Sulaiman, W. T., Extension on absolute summability factors of infinite series, *J. Math. Anal. Appl.*, *322*, 1224-1230 (2006).
8. Sulaiman, W. T., On some summability factors of infinite series, *Proc. Amer. Math. Soc.*, *115*, 313-317 (1992).