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On Simple Symmetric Random Walk in d-Dimensional Integer Lattice

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Abstract

This paper analyzes a simple symmetric random walk with finite steps in d-dimensional integer lattice, $\mathbb{Z}^d$ and introduces one of its applications. It focuses on the total number of ways in which the walk can be accomplished. The number of ways of accomplishment is used to find the probabilities associated with all possible outcomes as a generalization of the probability associated with return to origin. In addition, the paper discusses on the total number of possible outcomes. (Since the walk is executed in $\mathbb{Z}^d$, all the outcomes are integer points.) It provides an insight into the distribution of the integer lattice, $\mathbb{Z}^d$.

Keywords : allowed outcomes, fundamental integer lattice, initial condition, probability distribution function, and simple symmetric random walk.

Subject Classification Code (2010): 03G10, 60G50, 82B41.

1. Introduction

The theory of random walk studies stochastic processes formed by the successive summation of independent and identically distributed random variables[7]. A simple random walk is a walk that is characterized by a fixed size of each random variable, step. Moreover, the direction is random[12]. If the walk is equally probable in all possible direction, it is symmetric[8]. The purpose of this paper is to study a simple symmetric random walk with a finite number of steps, each of unit size, in d-dimensional integer lattice, $\mathbb{Z}^d$. Here, the integer lattice, $\mathbb{Z}^d$, refers to a set of integer points

$$x = (x^1, x^2, \dots, x^d),\ x^i = \text{ integer for } i = 1, 2, \dots, d,$$

in d-dimensional space, $\mathbb{Z}^d$ [4].

The study of simple symmetric random walk in d-dimensional integer lattice, $\mathbb{Z}^d$, deals with ideas like return to the origin, Bernoulli walk, range of random walk, recurrence, transience etc. [6-10]. In this paper, our focus is on possible outcomes of the walk and future outcomes of the simple symmetric random walk with a finite number of steps. Further, we discuss on the probabilities associated with each of them. This generalizes the probability distribution function of the walk from return to the origin to the occurrence of every outcome. It is noticed that the probability distribution function in 1-dimensional integer lattice, $\mathbb{Z}$, reduces to Bernoulli distribution. Finally author uses the concepts of length, concepts of equations of circles, spheres, and the like in taxicab geometry[1-2] to describe the distribution of integer points in -dimensional integer lattice, $\mathbb{Z}^d$.

2. Definitions and Notations :

Before getting into the details of the main propositions, some definitions need to be addressed.

In this paper, much of our discussion will revolve around the simple symmetric random walk. Therefore, throughout when a walk is referred, it means mean a simple symmetric random walk unless otherwise noted. Moreover, the size of a step is considered to be one unit.

Definition 2.1. The number of steps assigned for a walk is called initial condition. $|n|$ represents the initial condition. If n is negative, how a walk was accomplished is predicted. In contrast, if is positive, how a walk will be accomplished is predicted. For convenience, it is assumed that the walk starts at the origin when the future is predicted and that it terminates at the origin when the past is predicted.

Definition 2.2. When the initial condition of a walk that is accomplished is known, the integer points where the walk started can be predicted. These outcomes are called allowed past outcomes. Similarly, when the initial condition of walk that is to be accomplished is known, the possible integer points where the walk will terminate can be predicted. These integer points are called allowed future outcomes.

$S_{|n|} = (x^1, x^2, \dots, x^d)$ represents an allowed outcome of a walk with an initial condition, $|n|$ in $\mathbb{Z}^d$. The superscript of $x^i, (i = 1, 2, \dots, d)$ represents that x^i steps are mandatorily required (from the origin) to be executed in an i^{th} direction (for the occurrence of discussed outcome). Moreover, $\left|\{S_{|n|}\}\right|_d$ represents the number of allowed outcomes of the walk.

$P(S_{|n|} = (x^1, x^2, \dots, x^d))$ represents the probability of occurrence of the outcome, $S_{|n|} = (x^1, x^2, \dots, x^d)$, of a walk with an initial condition, $|n|$ in $\mathbb{Z}^d$. It is a measure that describes how probable the allowed future outcome or allowed past outcome is accordingly as n is positive or negative.

Definition 2.3. The length between two integer points, $x = (x^1, x^2, \dots, x^d)$ and $y = (y^1, y^2, \dots, y^d)$ in taxicab geometry is, $dist_T(x, y) = \sum_{i=1}^{d} |x^i - y^i|$ [1-2]. It is the most minimum number of steps, each of unit size, required to reach y from x or vice versa. Using this definition of length, a concept of oddness and evenness, parity, of an integer point in taxicab geometry.

The parity of an integer point $(x^1, x^2, \dots, x^d)$ is defined as oddness or evenness of the length between the integer point and the origin, $(0, 0, \dots, up\ to\ d\ terms)$ in taxicab geometry. Here, $dist_T(x, 0) = \sum_{i=1}^{d} |x^i - 0|$ describes the parity of the point, $(x^1, x^2, \dots, x^d)$, so that it is called the parity number.

Definition 2.4. The most extreme integer points, $(x^1, x^2, \dots, x^d)$, among the allowed outcomes of a walk are called boundaries. Broadly, those allowed outcomes which require $\sum_{i=1}^{d} |x^i| = |n|$ number of steps (from the origin) of a walk with an initial condition, $|n|$ in $\mathbb{Z}^d$ are called boundaries [Definition 2.3]. $B_{|n|} = (x^1, x^2, \ldots, x^d)$ represents boundaries of the walk. $\left|\{B_{|n|}\}\right|_d$ represents the number of boundaries of the walk with an initial condition, $|n|$ in $\mathbb{Z}^d$.

Up to this point, some basic terms and some notations have been introduced. In the following section, a lemma to prove a proposition about the probability distribution function for all allowed outcomes of a walk with the initial condition, $|n|$ in $\mathbb{Z}^d$ is introduced. Then, the number of allowed outcomes of the walk is dealt.

1. Propositions :

Lemma 3.1. The total number of possible ways for the accomplishment of a walk with an initial condition, $|n|$, *in* $\mathbb{Z}^d$, *is* $(2d)^{|n|}$.

Proof: There are $2d$ degrees of freedom of direction at the initial point of the walk in $\mathbb{Z}^d$. It is the same case after the execution of each step[6]. Using the multiplicative principle of counting, the number of ways that a walk with an initial condition, $|n|$, can be executed is $2d \times 2d \times \ldots upto\ |n|\ terms$. Thus, there are $(2d)^{|n|}$ ways for the accomplishment of the walk.

Proposition 3.2. The probability of occurrence of an allowed outcome, $S_{|n|} = (x^1, x^2, \ldots, x^d)$, *of a walk with an initial condition,* $|n|$, *in* $\mathbb{Z}^d$, *is*

$$\left(S_{|n|} = (x^1, x^2, \ldots, x^d)\right) = \frac{1}{(2d)^{|n|}}\left\{\sum \frac{(|n|)!}{(|x^1|+r_1)!r_1!(|x^2|+r_2)!r_2!\ldots(|x^d|+r_d)!r_d!}\right\}.$$

Here $(r_1, r_2, \ldots, r_d)$ *is a* d*-tuple such that* $\sum_{i=1}^{d} r_i = \frac{|n|-\sum_{i=1}^{d}|x^i|}{2}$ *and* r_i *is a non-negative integer.*

Proof: To find the probability of occurrence of an allowed outcome, we need to discuss the number of ways of a walk that results in the outcome. Firstly let us describe the direction of execution of each step. The sequence of execution is random, however. Finally, let us use combinatorics to calculate the number of contributing ways.

(For our positive n [Definition 2.1] So, the absolute sign is dropped down for now.) Let $S_n = (x^1, x^2, \ldots, x^d)$ be an allowed outcome of a walk with an initial condition, n, in $\mathbb{Z}^d$. The outcome mandatorily requires a length of $dist_T(x, 0) = \sum_{i=1}^{d}|x^i - 0|$ to be described [Definition 2.3.]. It means $\sum_{i=1}^{d}|x^i|$ number of steps: x^i in an i^{th} direction is executed for the outcome to occur [Definition 2.2.]—[Definition 2.3.].

Rest of the steps, $\left(n - \sum_{i=1}^{d}|x^i|\right)$, should be executed so that there is an equalization[5] (It gives a sense that execution of rest of the steps should not contribute the walk extra steps so that the outcome does not differ from $S_n = (x^1, x^2, \ldots, x^d)$). For equalization, there should be an equal number of steps executed in both positive and negative direction of any of d directions. (It implies that the number of remaining steps has even parity. Because the difference between n and $\sum_{i=1}^{d}|x^i|$ is even, the parity of initial condition, n, is same to that of the allowed outcome [Definition 2.3.]) The freedom in choosing any of d directions make us form d-tuple of the number of steps out of $\frac{n - \sum_{i=1}^{d}|x^i|}{2}$ steps executed each in positive directions and negative directions selected randomly.

Let $(r_1, r_2, \ldots, r_d)$ be one of such d-tuples that r_i $(i = 1, 2, \ldots, d)$ is executed each in i^{th} positive direction and negative direction. Because the sum of the number of steps executed in positive and negative direction should be equal to the number of remaining steps, $\sum_{i=1}^{d} r_i = \frac{n-\sum_{i=1}^{d}|x^i|}{2}$. If x^i is non-negative, $(|x^i| + r_i)$ steps are executed in an i^{th} positive direction and r_i in the i^{th} negative direction. If x^i is negative, $(|x^i| + r_i)$ steps are executed in the i^{th} negative direction. In contrast, if executed in an i^{th} positive direction. Now, the directions of execution of each step have been described. A similar discussion can be done with different ordered sets of $(r_1, r_2,..., r_d)$ that can be formed by the constraint, $\sum_{i=1}^{d} r_i = \frac{n-\sum_{i=1}^{d}|x^i|}{2}$.

The number of ordered execution of n steps along different directions as discussed is $\sum \frac{n!}{(|x^1|+r_1)!r_1!(|x^2|+r_2)!r_2!\dots(|x^d|+r_d)!r_d!}$. Here, the d-tuple $(r_1, r_2,..., r_d)$ can be varied with the help of the constraint, $\sum_{i=1}^{d} r_i = \frac{n-\sum_{i=1}^{d}|x^i|}{2}$. So, summation, $\sum$, is required. A similar discussion that deals with the past can be done when n is negative, except that $|n|$ replaces n.

The number of ordered execution of $|n|$ steps along different directions, $\sum \frac{|n|!}{(|x^1| + r_1)!\, r_1!\, (|x^2| + r_2)!\, r_2! \dots (|x^d| + r_d)!\, r_d!}$, is the number of ways that contribute the walk for the occurrence of the allowed outcome, $S_{|n|} = (x^1, x^2, \dots, x^d)$. Because there are $(2d)^{|n|}$ ways for accomplishment of the walk [Lemma 3.1.], the probability that the allowed outcome, $S_{|n|} = (x^1, x^2, \dots, x^d)$, occurs is

$$P\left(S_{|n|} = (x^1, x^2, \dots, x^d)\right) = \frac{1}{(2d)^{|n|}}\left\{\sum \frac{(|n|)!}{(|x^1| + r_1)!\, r_1!\, (|x^2| + r_2)!\, r_2! \dots (|x^d| + r_d)!\, r_d!}\right\} \quad (3.2a)$$

Here $(r_1, r_2,..., r_d)$ is a d-tuple such that r_i is a non-negative integer and $\sum_{i=1}^{d} r_i = \frac{|n|-\sum_{i=1}^{d}|x^i|}{2}$.

In eq. 3.2.a., $S_{|n|} = (x^1, x^2, \dots, x^d)$ is actually an arbitrary allowed outcome of a walk. Thus, it is the probability distribution function for a walk with an initial condition, $|n|$ in $\mathbb{Z}^d$

Corollary 3.2.1. In $\mathbb{Z}$, the constraint $\sum_{i=1}^{d} r_i = \frac{|n|-\sum_{i=1}^{d}|x^i|}{2}$ reduces to $r_1 = \frac{|n|-|x^1|}{2}$, and $S_{|n|} = (x^1, x^2, \dots, x^d)$ reduces to $S_{|n|} = (x^1)$. Thus, the probability distribution function reduces to

$$P\left(S_{|n|} = (x^1)\right) = \frac{1}{(2)^{|n|}} \frac{(|n|)!}{\left(\frac{|n|+|x^1|}{2}\right)!\left(\frac{|n|-|x^1|}{2}\right)!}. \quad (3.2.1.a.)$$

This is the Bernoulli distribution function[10]. When the initial condition is $2|n|$ and $S_{2|n|} = (0)$, eq. 3.2.1.a. reduces to

$$P\left(S_{2|n|} = (0)\right) = \frac{1}{(2)^{2|n|}}\binom{2|n|}{|n|}. \quad (3.2.1.b.)$$

This is the probability of return to the origin[5].

Lemma 3.3. The total number of allowed outcomes of a walk with an initial condition, $|n|$, in $\mathbb{Z}$ is

$$\left|\{S_{|n|}\}\right|_1 = x + 1$$

Proof: The number of allowed outcomes of a walk is constrained by the initial condition, $|n|$. Proposition 3.2 (par. 3) discusses that the initial condition and the allowed outcomes have same parity. So, the set of allowed outcomes in one dimensional integer lattice, $\mathbb{Z}$, is $\{(-|n|), (-|n| + 2), \dots, (|n| - 2), (|n|)\}$[12].

When $|n| = 1$, the set of allowed outcomes is $\{S_1\} = \{(-1), (1)\}$. So, $|\{S_1\}|_1 = 2 = 1+1$.

When $|n| = 2$, the set of allowed outcomes is $\{S_2\} = \{(-2), (0), (2)\}$. So, $|\{S_2\}|_1 = 3 = 2+1$.

Let the proposition be true when $|n| = k$. The set of allowed outcomes is $\{S_k\} = \{(-k), (-k+2), ..., (k-2), (k)\}$. Moreover, $|\{S_k\}|_1 = k + 1$.

Since the initial condition and the allowed outcomes have same parity, and because the supposition that the proposition is true when $|n| = k$, is needed to be used, let us check whether it is true when $|n| = k + 2$. The set of allowed outcomes is $\{S_{k+2}\} = \{(-k-2), (-k), (-k+2), ..., (k-2), (k), (k+2)\}$. There are two allowed outcomes: $(-k-2)$ and $(k+2)$ more than that of $\{S_k\}$. It is obvious that $|\{S_{k+2}\}|_1 = |\{S_k\}|_1 + 2 = (k+1) + 2 = (k+2) + 1$, which is true.

So, for a walk with an initial condition, $|n|$, *in* $\mathbb{Z}$, the total number of allowed outcomes is

$$\left|\{S_{|n|}\}\right|_1 = x + 1. \tag{3.3.a.}$$

Lemma 3.4. The total number of boundaries of a walk with an initial condition, $|n|$, *in* $\mathbb{Z}^d$ *is*

$$\left|\{B_{|n|}\}\right|_d = \left|\{S_{|n|}\}\right|_d - \left|\{S_{|n|-2}\}\right|_d.$$

Proof: The number and parity of allowed outcomes are constrained by the initial condition, so the allowed outcomes of a walk with an initial condition, $|n|$, and a walk with an initial condition, $|n| - 2$, have same parity.

If 2 steps out of $|n|$ steps undergo equalization, the former walk seems to be the later walk. It can be inferred that the allowed outcomes of a walk with an initial condition $|n| - 2$ are the allowed outcomes of a walk with an initial condition $|n|$.

In contrast, if the discussed 2 steps do not undergo equalization, and contribute 2 steps to each allowed outcome of the walk (that seemed to be a walk) with the initial condition $|n| - 2$, it results in the boundaries of the walk with initial condition $|n|$ as allowed outcomes. It is obvious that for a walk with an initial condition, $|n|$, in $\mathbb{Z}^d$, we get the total number of boundaries when the total number of the allowed outcomes of the former walk, is subtracted from the total number of the allowed outcomes of the latter one. That is,

$$\left|\{B_{|n|}\}\right|_d = \left|\{S_{|n|}\}\right|_d - \left|\{S_{|n|-2}\}\right|_d. \tag{3.4.a.}$$

Note: The formula holds good for all nonzero integers because the distribution of the boundaries, $B_{|n|} = (x^1, x^2, \dots, x^d)$, is similar for any initial condition. The distribution is described by the equation,

$$\textstyle\sum_{i=1}^{d}|x^i| = |n|. \tag{3.4.b.}$$

Because a walk with an initial condition, 0, is certain to be at the initial position[8], it is not actually a random walk. That is the formula does not hold good for the initial condition, 0, however.

Here eq. 3.4.b. represents circles, spheres, and hyperspheres centered at origin and radius $|n|$ in two-dimensional taxicab geometry, three-dimensional taxicab geometry and higher dimensional taxicab geometry respectively[3-9-11]. It means eq. 3.4.b. describes the distribution of integer points with same parity number [Definition 2.3.].

Corollary 3.4.1. There are two boundaries of a walk in $\mathbb{Z}$. *We can calculate using* eq. 3.3.a *when* $|n| = k$ *and when* $|n| = k - 2$ *and subtracting the latter one from the former one to get two boundaries.*

It is difficult to calculate the number of allowed outcomes by considering only the facts that the walk is executed in $\mathbb{Z}^d$ and the initial condition is $|n|$. We have in our hand the total number of allowed outcomes of a walk with a finite initial condition in $\mathbb{Z}$. This information can be used to calculate the number of allowed outcomes of a walk in $\mathbb{Z}^d$. The calculation is recursive which means the numbers of allowed outcomes in lower dimensional integer lattices are required to calculate the number of allowed outcomes in higher dimensional integer lattice.

The basic concept of calculating the number of allowed outcomes of a walk with the initial condition, $|n|$, in $\mathbb{Z}^d$ is: Let us consider an f-dimensional integer lattice, $\mathbb{Z}^f$, is less than d, and perform the walk. An integer point $(x^1, x^2, \dots, x^f)$ in $\mathbb{Z}^f$ mandatorily requires $\sum_{i=1}^{f}|x^i|$ number of steps [Definition 2.3.], so, the point, $(x^1, x^2, \dots, x^d)$, is constrained by $\sum_{i=1}^{f}|x^i| \le |n|$. If there are remaining steps, they can be used as initial condition for a walk in an extra $(f-d)$-dimensional integer lattice, $\mathbb{Z}^{f-d}$. We can find the total number of allowed outcomes of this walk (It is assumed that the formula to calculate the number allowed outcomes in $\mathbb{Z}^{f-d}$ and in $\mathbb{Z}^f$ are known) and sum up all the allowed outcomes contributed by the integer points that is constrained by $\sum_{i=1}^{f}|x^i| \le |n|$.

Definition 3.5. Let us consider a walk with an initial condition, $|n|$, in $\mathbb{Z}^d$. An f-dimensional integer lattice, $\mathbb{Z}^f$, f is less than d, integer points of which we consider as originas for further walk in an extra $(f-d)$-dimensional integer lattice, $\mathbb{Z}^{f-d}$, to calculate the total number of allowed outcomes of the walk in $\mathbb{Z}^d$ is called fundamental integer lattice.

Proposition 3.6. The total number of allowed outcomes of a walk with an initial condition, $|n|$, in $\mathbb{Z}^d$ is,

$$\left|\{S_{|n|}\}\right|_d = \left|\{S_{|n|}\}\right|_{d-f} + \sum_{i=1}^{|n|}\left(|\{B_i\}|_f \left|\{S_{|n|-i}\}\right|_{d-f}\right)$$

Here, $\mathbb{Z}^f$ is a fundamental integer lattice, and f is less than d. It is assumed that $\left|\{S_{|n|}\}\right|_{d-f}$ and $|\{B_i\}|_f$ are known.

Proof: Let us consider a walk with an initial condition, $|n|$, in $\mathbb{Z}^f$. Let the walk after execution of $\sum_{i=1}^{f}|x^i|$ number of steps without equalization be at $B_{\left(\sum_{i=1}^{f}|x^i|\right)} = (x^1, x^2, \dots, x^f)$ [Definition 2.4.]. Each of the integer points in $\mathbb{Z}^f$ has an extra $(f-d)$-dimensional integer lattice, $\mathbb{Z}^{f-d}$, for execution of $\left(|n| - \sum_{i=1}^{f}|x^i|\right)$ steps, and can contribute $\left|\left\{S_{\left(|n|-\sum_{i=1}^{f}|x^i|\right)}\right\}\right|_{d-f}$ allowed outcomes. The total contribution made by integer points represented by $B_{\left(\sum_{i=1}^{f}|x^i|\right)} = (x^1, x^2, \dots, x^f)$ in $\mathbb{Z}^{f-d}$ is $\left|\left\{B_{\left(\sum_{i=1}^{f}|x^i|\right)}\right\}\right|_f \left|\left\{S_{\left(|n|-\sum_{i=1}^{f}|x^i|\right)}\right\}\right|_{d-f}$ But $\sum_{i=1}^{f}|x^i|$ can be varied by the constraint $\sum_{i=1}^{f}|x^i| \le |n|$. Because B_0 does not hold good [Lemma 3.4.], $1 \le \sum_{i=1}^{f}|x^i| \le |n|$. Thus, the total number of allowed outcomes contributed by the discussed integer points of $\mathbb{Z}^{f-d}$ (except the origin of $\mathbb{Z}^{f-d}$) is

$$\sum_{i=1}^{|n|}\left(|\{B_i\}|_f \left|\{S_{|n|-i}\}\right|_{d-f}\right) \tag{3.6.a}$$

The walk at the origin of $\mathbb{Z}^f$ has $|n|$ steps that can be executed in $\mathbb{Z}^{f-d}$. So, the number of allowed outcomes contributed by the origin is

$$\left|\{S_{|n|}\}\right|_{d-f} \quad (3.6.b)$$

Finally, the total number of allowed outcomes of a walk with initial condition $|n|$, in $\mathbb{Z}^{f-d}$ is the sum of eq. 3.6.a. and eq. 3.6.b. So,

$$\left|\{S_{|n|}\}\right|_{d} = \left|\{S_{|n|}\}\right|_{d-f} + \sum_{i=1}^{|n|}\left(|\{B_i\}|_f \left|\{S_{|n|-i}\}\right|_{d-f}\right) \quad (3.6.c)$$

Where $\mathbb{Z}^f$ is a fundamental integer lattice, and f is less than d.

Corollary 3.6.1. The total number of allowed outcomes of a walk with initial condition, $|n|$, in $\mathbb{Z}^2$, is,

$$\left|\{S_{|n|}\}\right|_2 = (|n| + 1)^2.$$

Proof: Let the fundamental space be $\mathbb{Z}$ such that $|\{B_i\}|_1 = 2$ [Corollary 3.4.1], $|\{S_{|n|-i})\}|_1 = (|n| - i + 1)$ and $|\{S_{|n|})\}|_1 = (|n| + 1)$ [Lemma 3.3.]. Now using these quantities in eq. 3.6.c., we get,

$$\left|\{S_{|n|}\}\right|_2 = (|n| + 1) + \sum_{i=1}^{|n|} 2(|n| - i + 1)$$

$$\left|\{S_{|n|}\}\right|_2 = (|n| + 1)^2. \quad (3.6.1.a.)$$

This method can be used similarly to calculate the number of allowed outcomes of a walk in higher dimensional integer lattice, $\mathbb{Z}^d$.

4. Conclusion and Future Work

The paper analyzed a simple symmetric random walk of finite steps in $\mathbb{Z}^d$ and got some generalized results. It broadened the probability distribution function of the walk from the return to origin to all the possible outcomes of the walk. Moreover, it discussed the number of allowed outcomes. It gave an insight into the number of integer points $(x^1, x^2, \dots, x^d)$ having the same parity, or having the same parity number. The equation $\sum_{i=1}^{d}|x^i| = |n|$ explained the distribution of integer points in $\mathbb{Z}^d$.

One direction for the future work could be to explore the properties of circles, spheres and the like in taxicab geometry. In this paper, only the integer lattice is considered for the random walk. It can be generalized to other ordered lattices, however. In addition outcomes of a walk with infinite steps as initial condition along with the probability associated with them can be studied.

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